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RC-gyrator ladder and RLC symmetrical lattice network synthesis - A state-space approach

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RC-GYRATOR LADDER AND RLC SYMMETRICAL LATTICE NETWORK SYNTHESIS - A STATE-SPACE APPROACH

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ABSTRACT

This dissertation presents two network realization procedures using the state-space approach. The first procedure is the realization of a transfer function. A state model (A,B,C,D) is obtained from the given transfer function by inspection, where A is a companion matrix. Through a similarity transformation T, another state model (F,G,H,J) is obtained, where F is a tridiagonal matrix and can be realized by a RC-gyrator ladder network. The input source is inserted in the proper position of the ladder network, and the output is obtained through a summing circuit. It is a unified procedure and uses simple algebraic computation. A general expression of the similarity transformation T is derived for any order. It is a minimal realization.

The second procedure is the realization of an A matrix with a symmetrical lattice network. It is shown that the network functions of a symmetrical lattice network have a common factor in their numerators and denominators. Also, the eigenvalues of the A matrix are the poles and zeros of the driving-point functions. The A-matrix realization procedure is based on these properties. The synthesized network that consists of RC, RL, LC, or RLC elements depends on the locations of the eigenvalues of A. It is not a minimal realization.

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I. INTRODUCTION

A. STATEMENT OF THE PROBLEM

This dissertation is divided into two parts. The first part presents a state-space transfer-function synthesis with resistors, capacitors, gyrators, and operational amplifiers. It is included in Chapter 3 and Chapter 4. The second part is included in Chapter 5, and it presents an A-matrix realization procedure using symmetrical lattice networks. The motivation is given below.

It is known that control system theory and network theory have been closely related, because the mathematical methods in both fields are very similar, or identical. However, the recent developments of state-space methods in control system theory have been more advanced than those in network theory (1). The modern approach in system theory uses state-space methods while the classical approach uses frequency domain or Laplace-transform methods. Some distinct differences between these two approaches are:

1) The modern approach uses a state-space description of a network or system which emphasizes the internal structure as well as the input-output performance. The classical approach uses a Laplace-transform description which emphasizes the input-output performance only.

2) The modern approach uses matrix algebra as a mathematical tool which is easily formulated and programmable on a digital computer. The classical approach is

more concerned with complex variable analysis and relies on a variety of synthesis techniques.

Another interesting feature of the state-space method is that it can be used to solve the stability and minimization problems and has resulted in some new concepts such as controllability and observability.

Synthesis of a transfer function is a primary consideration in network theory and in control system theory. The network realization of transfer functions has been presented by many authors (1-9). But a simple unified state-space synthesis procedure using passive networks without transformers and with a minimum number of reactive elements has not yet been found.

The first part of this research presents a systematic state-space synthesis procedure for various kinds of transfer functions. The synthesized network contains a minimum number of reactive elements. The gyrators and the capacitors are used, for they can replace inductors and transformers, which are undesirable in practical applications because of their weight, size, and fabrication difficulties. It is impossible to fabricate inductors and transformers in integrated circuits, while an ideal gyrator can be fabricated in an integrated circuit as an individual component or realized by operational amplifiers (11,12).

The second part presents an A-matrix realization procedure with symmetrical lattice networks. The symmetrical lattice networks are investigated in this research because of their symmetrical structure and because their transfer functions can have right-half s-plane transmission zeros (10). The motivation is to develope a unified procedure for realizing a state model with symmetrical lattice networks without transformers. The following section describes in detail the technical approach.

B. TECHNICAL APPROACH

The state model of a linear passive network has the following general form,

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) \tag{1.B.1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1.B.1b}$$

where x is the state vector consisting of twig-capacitor voltages and chord-inductor currents, u is the input source vector, and y is the output vector. Equation (1.B.1a) is called the state equation. Equation (1.B.1b) is called the output equation. It is noted that A,B,C, and D are constant matrices which contain the element values and the topological information including the input source and the output. Of these matrices, A is the most important one, because it contains the complete description of the network element interconnections and values. It also describes the natural response of a network.

In the classical approach the driving-point and the transfer functions may be used for the port descriptions of networks. These network functions are rational functions of the complex variable s, with a certain distribution of poles and zeros. They completely determine the external behavior of the network under any excitation, but they give no information about the internal structures. The network transfer function W(s) can be obtained from equations (1.B.1) by taking their Laplace transform. Thus,

$$W(s) = Y(s)/U(s)$$

$$= C(sI-A)^{-1}B + D \qquad (1.B.2)$$

where I is an identity matrix. In equation (1.B.2) B,C, and D are constant matrices; therefore, the eigenvalues of A are the poles of the transfer function W(s) which are also called the natural frequencies of the network (7). Thus the A matrix completely describes the natural response and contains the internal structure information.

The first part of this research starts by obtaining an A matrix in symbolic form. Since the A matrix contains all the network topological information and the element values, comparing a given numerical A matrix with the symbolic A matrix yields a source-free network realization. The realizable condition is also derived. The A matrix in tridiagonal form satisfies this condition and is used for the transfer-function synthesis.

To realize a transfer function, a necessary transformation is derived. This transformation transforms a state model, $R_1=(A,B,C,D)$, into another state model, $R_2=(F,G,H,J)$, where A is a companion matrix and F is a tridiagonal matrix. The reason is that R_1 can be obtained easily from W(s) by inspection, but R_2 is always realizable. The matrix F is realized with a RC-gyrator ladder network. The matrix G gives the information where to insert the input source. The output is obtained through a summing circuit which consists of an operational amplifier and some resistors whose values depend on the matrices H and J.

The second part presents an A-matrix realization procedure with symmetrical lattice networks. Some special properties relating the network functions and the eigenvalues of the A matrix are analyzed. The realization of a given A matrix is based on these properties. The realized network contains RC, RL, LC, or RLC elements depending on the locations of the eigenvalues on the s-plane.

II. REVIEW OF LITERATURE

Many classical techniques for transfer-function synthesis have been presented in the literatures. Karni (3), Yengst (4), Mitra (6) and many others have written texts for various classical synthesis procedures. Lucal (3,4) developed a procedure to realize driving-point and transfer functions as three-terminal RC networks. Guillemin (2,3,4) presented another synthesis procedure resulting in two-port networks. Fialkow and Gerst (2,3,4) showed the necessary conditions for a transfer function to be realizable. Ho (4) developed a matrix factorization technique resulting in RLC ladder networks. Weinberg (4,14) presented a procedure for transfer-function realization with symmetrical lattice networks. Shenoi (15) designed a transistor circuit to realize a gyrator and cascaded it between two RC networks. Most of the classical synthesis procedures are in the complex-frequency domain and usually vary for each of the different types of transfer functions.

Since Bashkow(16) first defined the A matrix as a new network description in 1957, there has been a growing interest in the state-model concept of network analysis and synthesis. Bryant (17) in 1962 derived the explicit form of Bashkow's A matrix for a RLC network.

The necessary and sufficient condition for a constant A matrix to be realizable by a passive time-invariant network is that A is a stable matrix, i.e., all the eigenvalues of A have nonpositive real parts and the purely imaginary eigenvalues have multiplicity one. Tow (18) and Silverman (19) presented different proofs for this condition with passive networks consisting of resistors, capacitors, and inductors or consisting of resistors, capacitors, and gyrators.

Many procedures have been presented in the realization of an A matrix. Dervisoglu (20) realized the A matrix with a class of RLC networks that contain no all-capacitor loops nor all-inductor cut-sets, and the resistive part forms a connected subgraph. Nordgreen and Tokad (21) used a different approach for realizing the A matrix with passive RLC networks. The given A matrix is partitioned and factored into

$$A = B^{-1}A_1$$

and the realizability depends on the property that A_1 must be a paramount matrix in order to be realized as a resistive network.

Yarlagadda (22) presented a procedure for realizing a tridiagonal A matrix with LC ladder network terminated with one resistor. This tridiagonal matrix is derived from the Routh array of a Hurwitz polynomial, and it is used in this research for the transfer-function synthesis. Yarlagadda and Ye (23) developed a procedure for A-matrix realization with RC, nullator, and norator networks. This procedure is then extended to realizing an admittance matrix.

The transfer-function synthesis using the state-space approach has been presented by several authors. The method by Dewild, Silverman, and Newcomb (24) is based upon the formation of an almost skew-symmetric admittance matrix Y loaded with capacitors. The matrix Y is made positive real by the use of Lyapunov transformation and is synthesized with resistors and gyrators. A minimum number of capacitors is used in the synthesized network.

Fowler and Yarlagadda (25) presented a unified state-space procedure for transfer-function synthesis. In this procedure a tridiagonal matrix is developed by using Navot's method (26). The synthesized network consists of RLC ladder networks in parallel and may require transformers for interconnection. Several books, such as those by Newcomb and Anderson(1,9), include synthesis of transfer function matrices using the state-space approach and are presently available.

Two different realization procedures are developed in this dissertation. The first procedure realizes a transfer function with a state model having a tridiagonal A matrix. The synthesized network consists a RC-gyrator ladder network and a summing circuit for the output. It is an interesting synthesis procedure from the modern system theory approach. The second procedure realizes a stable A matrix with RLC symmetrical lattice networks. This procedure uses a new property that relates the A-matrix eigenvalues to the critical frequencies of the driving-point immittance function.

III. RC-GYRATOR SYNTHESIS OF THE A MATRIX

A. INTRODUCTION

1. Outline

In Chapter 3 the A-matrix realization procedure is presented which is used in the transfer-function synthesis presented in Chapter 4. The procedure is outlined below.

1) In Section 3.B a homogeneous state equation in symbolic form for portless RC-gyrator networks is derived. Then the state equation is simplified under some topological constraint.

2) In Section 3.C an A-matrix realization procedure is presented. The realization of a tridiagonal A matrix is discussed in Section 3.C.4.

3) In Section 4.B a similarity transformation is presented. This transformation gives the desired state model for a given transfer function.

4) In Section 4.C a unified synthesis procedure for various kinds of transfer functions is presented.

The flow chart in Figure 3-1 gives more information about this synthesis procedure.

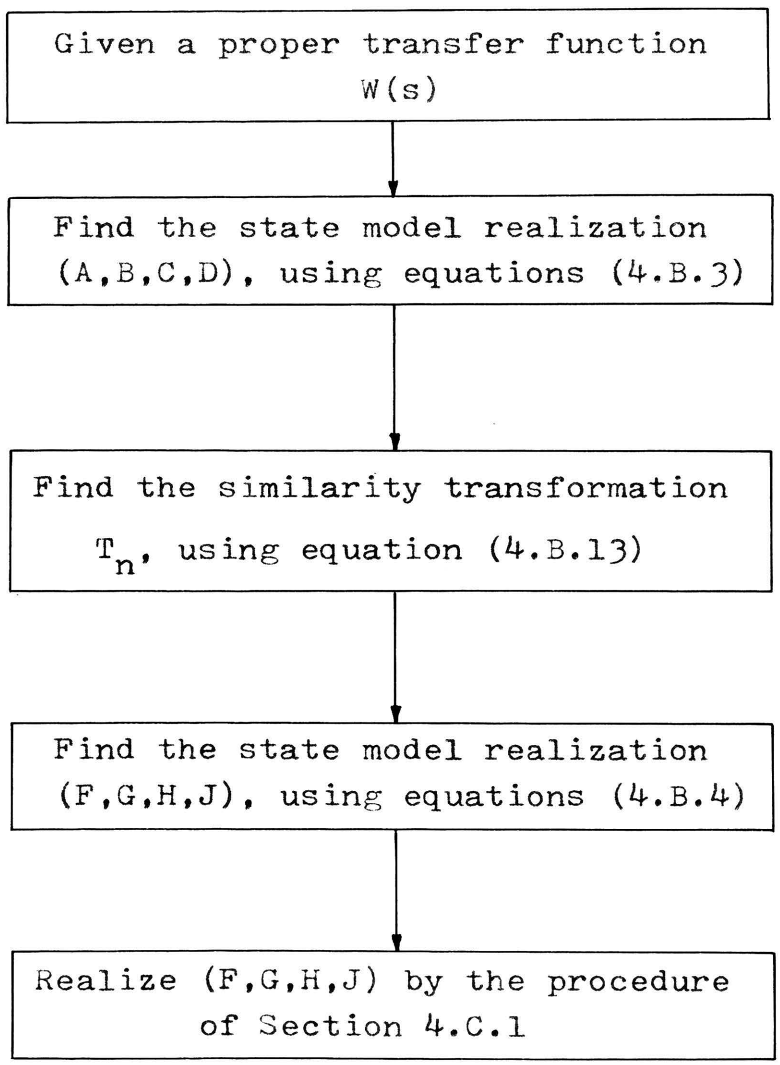


Figure 3-1. The flow chart for the procedure of transfer-function synthesis

2. Preliminary Considerations

A scalar transfer function $W(s)$ is defined as a real rational function which relates the Laplace-transformed input $U(s)$ to the output $Y(s)$ through

$$Y(s) = W(s)\,U(s) \tag{3.A.1}$$

and can be written as

$$W(s) = \frac{c_o s^m + c_1 s^{m-1} + \ldots + c_m}{s^n + a_1 s^{n-1} + \ldots + a_n} \quad ; \; m < n \tag{3.A.2}$$

If the denominator polynomial is Hurwitz and there is no common factor in the denominator and numerator of $W(s)$, $W(s)$ is called a proper transfer function.

$W(s)$ is related to its state model

$$\dot{x} = Ax + Bu \tag{3.A.3a}$$

$$y = Cx + Du \tag{3.A.3b}$$

by

$$W(s) = C(sI_k - A)^{-1}B + D \tag{3.A.4}$$

where A is a $(k \cdot k)$ matrix, B is a $(k \cdot 1)$ column vector, C is a $(1 \cdot k)$ row vector, D is a scalar constant, and k is the dimension of the state. The set of constant matrices, $R=(A,B,C,D)$, is called a state model realization of $W(s)$. It is known that R is not unique (1). To realize $W(s)$, it is necessary only to realize a set of matrices (A,B,C,D).

B. A STATE EQUATION OF RC-GYRATOR NETWORKS

1. The Linear Graph of Ideal Gyrators

An ideal gyrator is a two-port nonreciprocal device and can be described by the impedance matrix,

$$Z_G = \begin{bmatrix} 0 & -r \\ r & 0 \end{bmatrix} \tag{3.B.1}$$

or by the admittance matrix,

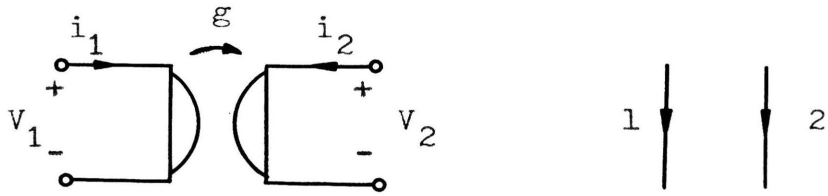
$$Y_G = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \tag{3.B.2}$$

as shown in Figure 3-2a. Both Z_g and Y_g are skew-symmetric matrices; r is the gyration resistance, and g is the gyration conductance. Its linear graph can be represented by two branches either in a tree or in a co-tree (7) as shown in Figure 3-2b. (See Appendix E)

The linear graph of a n-port gyrator network with an admittance matrix Y_G is represented by n branches either in a tree or in a co-tree. Consider a three-port two gyrator network whose admittance matrix is:

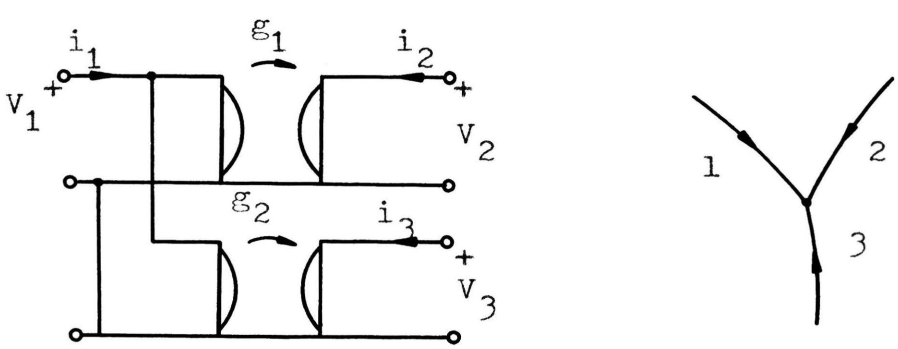
$$Y_G = \begin{bmatrix} 0 & g_1 & g_2 \\ -g_1 & 0 & 0 \\ -g_2 & 0 & 0 \end{bmatrix} \tag{3.B.3}$$

The gyrator network and its linear graph are shown in Figure 3-3.



(a) (b)

Figure 3-2. (a) An ideal gyrator and (b) its linear graph



(a) (b)

Figure 3-3. (a) A three-port gyrator network and (b) its linear graph

2. Derivation of The State Equation

In this section a homogeneous state equation of a RC-gyrator network is derived. The matrix elements used in this section are defined as follows:

I - An identity matrix.

Y_R - Diagonal resistor conductance matrix.

Y_G - Skew-symmetric gyrator admittance matrix.

C_t, R_t - Diagonal capacitance and resistance matrices.

B_t, B_l - Unimodular circuit submatrices. (See Appendix B)

Q_t, Q_l - Unimodular cut-set matrices.

V_{Rt}, I_{Rt} - Twig-resistor voltage vector and current vector.

V_{Ct}, I_{Ct} - Twig-capacitor voltage vector and current vector.

V_{Rl}, I_{Rl} - Link-resistor voltage vector and current vector.

V_{Gl}, I_{Gl} - Link-gyrator voltage vector and current vector.

B_{CG}, B_{RG}, B_{CR}, B_{RR} - Unimodular submatrices of a circuit matrix, where the first subscript indicates that the element is in the tree, the second, is in the co-tree.

The derivation starts with an arbitrary network containing resistors, capacitors, and gyrators. Find a proper tree; that is, all capacitors of the network are in the tree. Write the fundamental circuit equation in symbolic form (7),

$$B_f\ v(t) = 0 \qquad (3.B.4)$$

where

$$B_f = \begin{bmatrix} B_t & B_l \end{bmatrix} \tag{3.B.5}$$

the cut-set equation,

$$Q\ i(t) = 0 \tag{3.B.6}$$

where

$$Q = \begin{bmatrix} Q_t & Q_l \end{bmatrix} = \begin{bmatrix} I & -B_t' \end{bmatrix} \tag{3.B.7}$$

and the v-i relations of the network components. From these sets of equations a state equation can be derived. Since this network contains no sources, the state equation is homogeneous, i.e.,

$$\dot{x} = A\ x \tag{3.B.8}$$

The derivation is now presented in detail below.

Assume an arbitrary network N consisting of resistors, capacitors, and gyrators with no all-capacitor loops. Find a proper tree with all the gyrators contained in the co-tree and write the fundamental circuit equation as follows,

$$\begin{bmatrix} B_{CG} & B_{RG} & I & 0 \\ B_{CR} & B_{RR} & 0 & I \end{bmatrix} \begin{bmatrix} V_{Ct} \\ V_{Rt} \\ V_{Gl} \\ V_{Rl} \end{bmatrix} = 0 \tag{3.B.9}$$

where all capacitors are in the tree and all gyrators are in the co-tree.

From equations (3.B.6) and (3.B.7) the cut-set equation is

$$\begin{bmatrix} I & 0 & -B'_{CG} & -B'_{CR} \\ 0 & I & -B'_{RG} & -B'_{RR} \end{bmatrix} \begin{bmatrix} I_{Ct} \\ I_{Rt} \\ I_{Gl} \\ I_{Rl} \end{bmatrix} = 0 \qquad (3.B.10)$$

The v-i relations for the capacitors, tree-branch resistors, link resistors, and gyrators are

$$I_{Ct}(t) = C_t \frac{d}{dt} V_{Ct}(t) \qquad (3.B.11)$$

$$V_{Rt}(t) = R_t \, I_{Rt}(t) \qquad (3.B.12)$$

$$I_{Rl}(t) = Y_R \, V_{Rl}(t) \qquad (3.B.13)$$

and

$$I_{Gl}(t) = Y_G \, V_{Gl}(t) \qquad (3.B.14)$$

Now we proceed to obtain the state equation in the form of equation (3.B.8). Since network N contains only capacitors as energy-storage elements, the capacitor voltages are considered to be the state variables. From equations (3.B.10) and (3.B.11) we get

$$C_t \frac{d}{dt} V_{Ct} = B'_{CG} \, I_{Gl} + B'_{CR} \, I_{Rl} \qquad (3.B.15)$$

Substituting equations (3.B.13) and (3.B.14) into equation (3.B.15), we get

$$C_t \frac{d}{dt} V_{Ct} = B'_{CG} \, Y_G \, V_{Gl} + B'_{CR} \, Y_R \, V_{Rl}$$

Rewrite the right side of the above equation in matrix form

$$C_t \frac{d}{dt} V_{Ct} = \begin{bmatrix} B'_{CG}\ Y_G & B'_{CR}\ Y_R \end{bmatrix} \begin{bmatrix} V_{G1} \\ V_{R1} \end{bmatrix} \tag{3.B.16}$$

From equations (3.B.10) and (3.B.12) we get

$$\begin{aligned} V_{Rt} &= R_t \left[B'_{RG} I_{G1} + B'_{RR} I_{R1} \right] \\ &= R_t \left[B'_{RG} Y_G V_{G1} + B'_{RR} Y_R V_{R1} \right] \end{aligned} \tag{3.B.17}$$

The last step follows by substituting for I_{G1} and I_{R1} from equations (3.B.14) and (3.B.13). Substituting V_{Rt} from equation (3.B.17) into the first equation of (3.B.9), we get

$$B_{CG} V_{Ct} + B_{RG} R_t \left[B'_{RG} Y_G V_{G1} + B'_{RR} Y_R V_{R1} \right] + V_{G1} = 0$$

or

$$\left[I + B_{RG} R_t B'_{RG} Y_G \right] V_{G1} + \left[B_{RG} R_t B'_{RR} Y_R \right] V_{R1} = -B_{CG} V_{Ct} \tag{3.B.18}$$

From equations (3.B.9) and (3.B.17) we get

$$B_{CR} V_{Ct} + B_{RR} R_t \left[B'_{RG} Y_G V_{G1} + B'_{RR} Y_R V_{R1} \right] + V_{R1} = 0$$

and by rearranging it becomes

$$\left[B_{RR} R_t B'_{RG} Y_G \right] V_{G1} + \left[I + B_{RR} R_t B'_{RR} Y_R \right] V_{R1} = -B_{CR} V_{Ct} \tag{3.B.19}$$

Combine equations (3,B.18) and (3.B.19) in matrix form

$$\begin{bmatrix} I + B_{RG}R_tB'_{RG}Y_G & B_{RG}R_tB'_{RR}Y_R \\ B_{RR}R_tB'_{RG}Y_G & I + B_{RR}R_tB'_{RR}Y_R \end{bmatrix} \begin{bmatrix} V_{G1} \\ V_{R1} \end{bmatrix} = - \begin{bmatrix} B_{CG} \\ B_{CR} \end{bmatrix} V_{Ct} \tag{3.B.20}$$

Let the coefficient matrix be denoted by M

$$M = \begin{bmatrix} I + B_{RG}R_tB'_{RG}\ Y_G & B_{RG}R_tB'_{RR}Y_R \\ B_{RR}R_tB'_{RG}Y_G & I + B_{RR}R_tB'_{RR}Y_R \end{bmatrix} \tag{3.B.21}$$

and assume its inverse exists. Then the state equation can be obtained by substituting $(V_{G1}\ V_{R1})'$ from equation (3.B.20) into equation (3.B.16)

$$\dot{V}_{Ct} = -C_t^{-1} \begin{bmatrix} B'_{CG}Y_G & B'_{CR}Y_R \end{bmatrix} M^{-1} \begin{bmatrix} B_{CG} \\ B_{CR} \end{bmatrix} V_{Ct} \tag{3.B.22}$$

To assure the existence of the inverse of M and simplify the A matrix, a topological constraint is considered in the next section.

3. A Simplified State Equation

In order to obtain a simple closed form for the A matrix of equation (3.B.22) so that it can be used in the realization procedure, it is assumed that network N contains all its resistors in the co-tree. This implies that there are no tree-branch resistors; submatrices B_{RG} and B_{RR} are zero. The coefficient matrix M reduces to an identity matrix, and the state equation (3.B.22) now becomes

$$\dot{V}_{Ct} = - C_t^{-1} \begin{bmatrix} B'_{CG}Y_G & B'_{CR}Y_R \end{bmatrix} \begin{bmatrix} B_{CG} \\ B_{RG} \end{bmatrix} V_{Ct} \tag{3.B.23}$$

Then the A matrix has the form

$$A = - C_t^{-1} \left[B'_{CG}Y_GB_{CG} + B'_{CR}Y_RB_{CR} \right]$$

or

$$A = - C_t^{-1} \left[A_{sk} + A_{sy} \right] \tag{3.B.24}$$

where

$$A_{sk} = B'_{CG}Y_GB_{CG} \tag{3.B.25}$$

and

$$A_{sy} = B'_{CR}Y_RB_{CR} \tag{3.B.26}$$

Note that A_{sy} is a symmetric matrix and A_{sk} is a skew-symmetric matrix. The fundamental circuit matrix of equation (3.B.9) now becomes

$$B_f = \begin{bmatrix} B_{CG} & I & 0 \\ B_{CR} & 0 & I \end{bmatrix} \tag{3.B.27}$$

This section is now summarized. A source-free state equation has been derived as shown in equation (3.B.23) which corresponds to a class of RC-gyrator networks with the properties that there are no all-capacitor loops and all resistors are in the co-trees. When a numerical A matrix is compared with this symbolic A matrix, the element values and topological information can be obtained. The A-matrix realization procedure is presented in detail in the following section.

C. SYNTHESIS OF THE A MATRIX

1. Introduction

A transfer-function synthesis is equivalent to the realization of a set of matrices (A,B,C,D) which is the state model of the transfer function. The A matrix contains both the topological information and element value information. Therefore, its realization is fundamental to the transfer-function synthesis and is presented in this section. The matrices (B,C,D) which describe the input and output are considered in the next chapter.

In Section 3.C.2 the A-matrix realization procedure is outlined,and each step is discussed in detail.

In Section 3.C.3 an example is given which demonstrates the realization technique. The realization of an arbitrary A matrix depends on the paramount character of the matrix A_{sy} and on a proper extraction of the matrix C_t.

In Section 3.C.4 it is shown that a tridiagonal A matrix is always realizable by a ladder network. Thus it provides the motivation for finding a similarity transformation which transforms a companion A matrix into a tridiagonal matrix, since a companion A matrix is easily obtained from the denominator of the transfer function. This similarity transformation is included in Chapter 4.

2. A-Matrix Realization Procedure

The following procedure for realizing a real square A matrix is developed from the state equation (3.B.23). It should be pointed out that only those A matrices with the forms mentioned below and in the next section can be realized by this procedure. To realize an arbitrary stable A matrix, a certain condition must be satisfied as shown in the procedure.

Step 1) Extract or remove a diagonal matrix C_t^{-1} according to equation (3.B.24).

Step 2) Decompose the remaining part into a symmetric matrix A_{sy} and a skew-symmetric matrix A_{sk}.

Step 3) Check that A_{sy} is paramount. (See Appendix B).

Step 4) Decompose A_{sy} into $B_{CR}^{'}Y_R B_{CR}$ and A_{sk} into $B_{CG}^{'}Y_G B_{CG}$.

Step 5) Write the fundamental circuit matrix B_f and draw the corresponding network.

In Step 1 the matrix C_t gives the values of the capacitors and it is extracted in such a way that A_{sy} is made paramount. (See the example below).

In Step 2 the symmetric part and skew-symmetric part of a square matrix B are given by (29)

$$B_{sy} = (B + B')/2 \qquad (3.C.1a)$$

$$B_{sk} = (B - B')/2 \qquad (3.C.1b)$$

In Step 3 this method fails if A_{sy} is not paramount. In Step 4 the matrix A_{sy} is decomposed by Cederbaum's algorithm as shown in Appendix B. The matrix A_{sk} can always be decomposed by assuming B_{CG} is an identity matrix or a diagonal matrix with the diagonal elements one or zero where the zero entries correspond to the zero rows of A_{sk}. (See example below). The unimodular matrices B_{CR} and B_{CG} give the network topology . Y_R is a diagonal matrix and gives the values of the resistors. Y_G is a skew-symmetrical matrix which describes the port relations and gives the gyration conductances of the gyrators.

It should be pointed out that only those A matrices with A_{sy} paramount can be realized by this procedure. In the following section it is shown that a special tridiagonal A matrix which is obtained from the Routh array (22) can always be realized with a RC-gyrator ladder network, because the matrix A_{sy} is always paramount. Therefore, only this tridiagonal A matrix is considered for the transfer-function synthesis presented in Chapter 4.

It is interesting to point out that an A matrix of a passive RLC network is realizable by this procedure. For a RLC network the A matrix has the following general form,

$$A = - \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}^{-1} \begin{bmatrix} Y & -H \\ H' & Z \end{bmatrix} \qquad (3.C.2)$$

where Y and Z are paramount symmetrical matrices and C and L are positive-definite diagonal matrices (7,17). The symmetrical matrix A_{sy} of equation (3.C.2) is

$$A_{sy} = \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix}$$

which is paramount. The following example illustrates the realization procedure.

3. An Example

Consider the A matrix of a RLC network (13),

$$A = - \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & -1 & 1 \\ 0 & -2 & 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & -2 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \end{bmatrix} \qquad (3.C.3)$$

Step 1) Since this is a passive RLC A matrix, the matrix C_t can be extracted such that the off-diagonal elements which correspond to the matrix H in equation (3.C.2) are made skew-symmetric.

$$A = - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(3.C.4)

Thus, the diagonal capacitor matrix yields

$$C_1 = C_2 = 1f.$$

$$C_3 = C_4 = 1/2f.$$

$$C_5 = C_6 = 1/3f.$$

Step 2) Decompose the second matrix of equation (3.C.4) into A_{sy} and A_{sk}, according to equations (3.C.1).

$$A_{sy} =- \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{sk} =- \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 3) A_{sy} is a paramount matrix. It can be decomposed by Cederbaum's algorithm as shown below.

Step 4) Decompose A_{sy} into the triple product $B'_{CR}Y_RB_{CR}$.

$$A_{sy} = B'_{CR}\, Y_R\, B_{CR}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.C.5)$$

Equation (3.C.5) yields the unimodular matrix B_{CR} and the conductance matrix Y_R which gives the resistances for the four resistors of

$$R_1 = R_2 = R_3 = R_4 = 1 \text{ ohm.}$$

The matrix A_{sk} can be decomposed as follows.

$$A_{sk} = B'_{CG}\, Y_G\, B_{CG}$$

$$= \begin{bmatrix} 0 & & & & & \\ & 1 & & & \bigcirc & \\ & & 0 & & & \\ & & & 1 & & \\ & \bigcirc & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} & & & & 0 & 0 \\ & & \bigcirc & & -1 & 1 \\ & & & & 0 & 0 \\ & & & & 0 & -1 \\ 0 & 1 & 0 & 0 & & \\ 0 & -1 & 0 & 1 & & \bigcirc \end{bmatrix} \begin{bmatrix} 0 & & & & & \\ & 1 & & & \bigcirc & \\ & & 0 & & & \\ & & & 1 & & \\ & \bigcirc & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$$(3.C.6)$$

Equation (3.C.6) yields the unimodular matrix B_{CG} and the gyration admittance matrix Y_G which gives the gyration admittances for the three gyrators of $g_{25} = -1$, $g_{26} = +1$, and $g_{46} = -1$. The gyrator component network is shown in Figure 3-4a.

Step 5) From equation (3.B.27), the fundamental circuit matrix is

$$\left[\begin{array}{cccccc|cccccc|cccc}
0 & & & & & & 0 & & & & & & & & & \\
 & 1 & & & 0 & & & 1 & & & 0 & & & & & \\
 & & 0 & & & & & & 0 & & & & & 0 & & \\
 & & & 1 & & & & & & 1 & & & & & & \\
 & 0 & & & 1 & & & 0 & & & 1 & & & & & \\
 & & & & & 1 & & & & & & 1 & & & & \\
\hline
1 & -1 & 0 & 0 & 0 & 0 & & & & & & & 1 & & & \\
0 & 1 & -1 & 0 & 0 & 0 & & & & & & & & 1 & & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & & & 0 & & & & & & 1 & \\
1 & 0 & 0 & 0 & 0 & 0 & & & & & & & & 0 & & 1
\end{array}\right]$$

The network corresponding to this circuit matrix is shown in Figure 3-4b.

From the above example it is seen that the realization of an A matrix depends on the extraction of the matrix C_t such that the matrix A_{sy} is paramount. The following section shows that a special tridiagonal A matrix can always be realized by a RC gyrator ladder network.

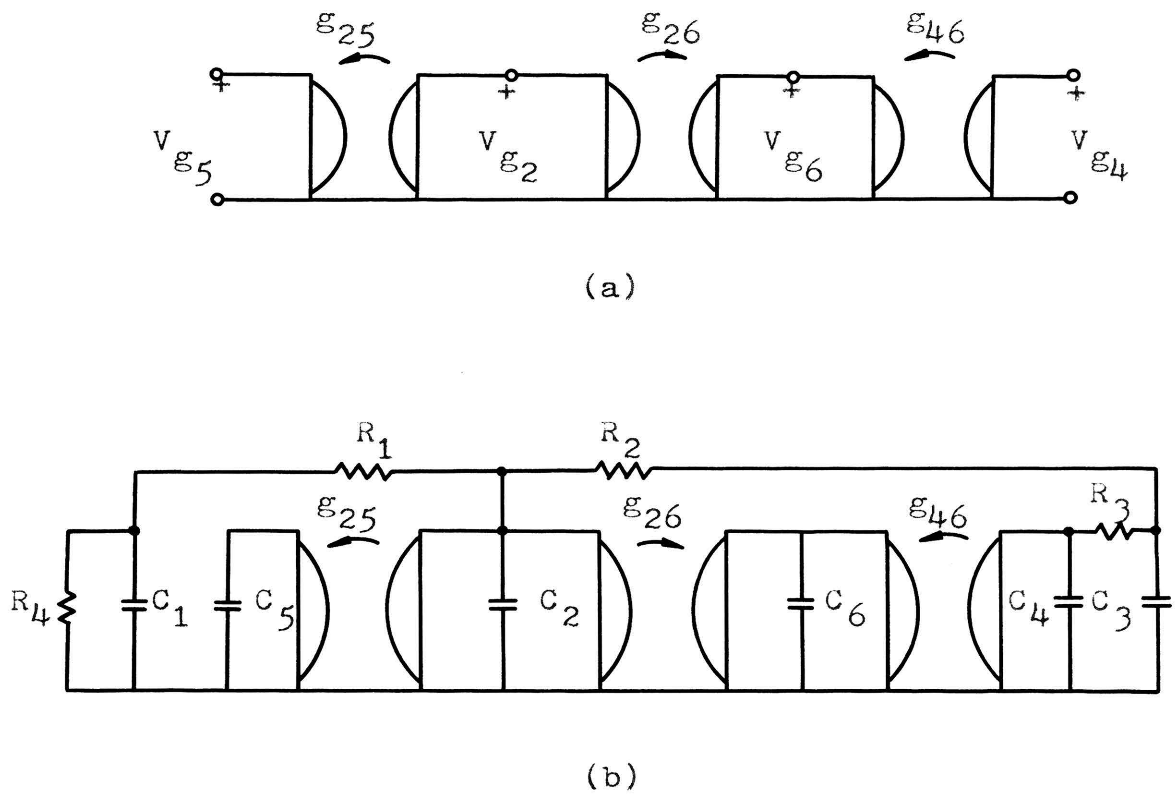


Figure 3-4. (a) The gyrator network and
(b) the synthesized network of Example (3.C.3)

4. Realization of a Tridiagonal A Matrix

It is known that a ladder network is related to a tridiagonal matrix. A tridiagonal A matrix can be derived from a Hurwitz polynomial by several methods. The modified Routh array method (22) gives only one nonzero element in the diagonal while Navot's method (26) gives two nonzero elements in the diagonal.

A tridiagonal matrix as obtained by the modified Routh array has the form

$$F = \begin{bmatrix} -k_1 & -k_2 & & & & \bigcirc & \\ k_2 & 0 & -k_3 & & & & \\ & k_3 & 0 & -k_4 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & \bigcirc & & & \ddots & \ddots & -k_n \\ & & & & & k_n & 0 \end{bmatrix} \qquad (3.C.7)$$

See Appendix C. When a tridiagonal A matrix with this form is realized by the procedure of Section 3.C.2, it is found that it can always be realized with a RC-gyrator ladder network as described below.

Step 1) The diagonal capacitor matrix C_t is simply an identity matrix which yields unity values for all of the n capacitors.

Step 2) The matrices A_{sy} and A_{sk} are easily obtained from equation (3.C.7).

$$A_{sy} = \begin{bmatrix} k_1 & & & \bigcirc \\ & 0 & & \\ \bigcirc & & \ddots & \\ & & & 0 \end{bmatrix}$$

$$A_{sk} = \begin{bmatrix} 0 & k_2 & & & \bigcirc \\ -k_2 & 0 & k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & k_n \\ \bigcirc & & & -k_n & 0 \end{bmatrix}$$

Step 3) A_{sy} is a paramount matrix.

Step 4) A_{sy} is decomposed as follows;

$$A_{sy} = B'_{CR}\ Y_R\ B_{CR}$$

$$= \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \begin{bmatrix} k_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

which gives the matrix B_{CR} and only one resistor with resistance of $R_1 = 1/k_1$.

A_{sk} is decomposed by letting $B_{CG} = I$, an identity matrix. Thus,

$$A_{sk} = B'_{CG}\ Y_G\ B_{CG} = Y_G$$

which yields (n-1) gyrators with admittances of k_2, k_3, ..., k_n.

Step 5)

$$B_f = \left[\begin{array}{cccc|cccc|c} 1 & & & \bigcirc & 1 & & & \bigcirc & 0 \\ & 1 & & & & 1 & & & 0 \\ & & \ddots & & & & \ddots & & \vdots \\ \bigcirc & & & 1 & \bigcirc & & & 1 & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 & \cdots & & 0 & 1 \end{array}\right]$$

The corresponding network is shown in Figure 3-5.

The same tridiagonal matrix realized by Yarlagadda's procedure results in an LC ladder network terminated with one resistor (22). Figure 3-5 is the fundamental circuit for the transfer-function synthesis presented in the following chapter. If the matrix F of equation (3.C.7) has a zero k_i, i.e., W(s) has a jw-axis simple pole, the ladder will be of two separate parts (22).

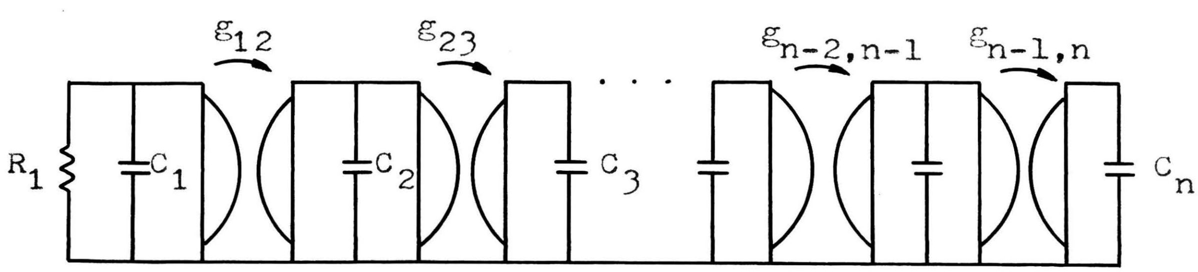


Figure 3-5. RC-gyrator ladder network of a tridiagonal A matrix

IV. RC-GYRATOR SYNTHESIS OF TRANSFER FUNCTIONS

A. INTRODUCTION

Chapter 3 presents the procedure for realizing an A matrix. It is shown in Section 3.C.4 that a tridiagonal matrix F can be realized with a RC-gyrator ladder network which is the fundamental circuit for the transfer-function synthesis presented in this chapter.

It is known that a state model (A,B,C,D) is easily obtained from a transfer function with A in companion form. A necessary similarity transformation T_n is derived in Section 4.B which transforms the state model (A,B,C,D) into a realizable state model (F,G,H,J), where F is a tridiagonal matrix. Then the state model (F,G,H,J) is realized by a procedure described in Section 4.C.

B. THE SIMILARITY TRANSFORMATION T_n

1. Transfer Function and Its Minimal Realizations

In this section an important theorem from system theory is first introduced. Based on this theorem a similarity transformation T_n is derived and is used for the synthesis of transfer functions presented in the next section.

A transfer function W(s) with no pole at infinity as defined in equation (3.A.2) can be written as

$$W(s) = d + \frac{c_1 s^{n-1} + \ldots + c_n}{s^n + a_1 s^{n-1} + \ldots + a_n} \tag{4.B.1}$$

or equivalently

$$W(s) = C(sI_k - A)^{-1}B + D \tag{4.B.2}$$

where (A,B,C,D) is a state-model realization of W(s).

<u>Minimal Realization</u>: The set of constant matrices R = (A,B, C,D) is said to be a minimal realization (9) or an irreducible realization of W(s), if k assumes its smallest possible value k_{min}, which is the minimum number of energy storage elements necessary to physically realize W(s).

It has been shown that all minimal realizations of a given transfer function W(s) have the same dimension k_{min}. Also the set of matrices (A,B,C,D) with matrix A in companion form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & & & 0 \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ \cdot & & & & & \cdot & \\ 0 & & & & & & 1 \\ -a_n & -a_{n-1} & \cdot & & \cdot & \cdot & -a_1 \end{bmatrix} \tag{4.B.3a}$$

$$B = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{4.B.3b}$$

$$C = \begin{bmatrix} c_n & c_{n-1} & \cdot\cdot\cdot & c_1 \end{bmatrix} \tag{4.B.3c}$$

$$D = d \tag{4.B.3d}$$

is a minimal realization of a transfer function W(s) where a necessary and sufficient condition is that W(s) has no common factors in its numerator and denominator (1,30). If this condition is satisfied, then the dimension k_{min} is equal to n and equals the order of the A matrix. The theorem for existence of T_n is now introduced.

<u>Theorem I</u>: Let $R_1 = (A,B,C,D)$ and $R_2 = (F,G,H,J)$ be two minimal realizations of a given transfer function W(s). Then there exists a nonsingular $(n \cdot n)$ matrix T_n,

such that

$$F = T_n^{-1} A T_n \qquad (4.B.4a)$$

$$G = T_n^{-1} B \qquad (4.B.4b)$$

$$H = C T_n \qquad (4.B.4c)$$

$$J = D \qquad (4.B.4d)$$

The proof of this theorem can be found in the reference (1). Note that the matrices (F,G,H,J) have the same dimensions as the matrices (A,B,C,D) respectively.

Based on Theorem I and Appendix C, a similarity transformation T_n is derived, which transforms the minimal realization $R_1 = (A,B,C,D)$ to another minimal realization $R_2 = (F,G,H,J)$, where R_1 is in the form of equations (4.B.3) and the matrix F in R_2 is in the tridiagonal form as shown in equation (3.C.7). A second-order case is first derived as an example to illustrate the procedure for obtaining the n^{th}-order similarity transformation.

2. A Second-Order Similarity Transformation T_2

Consider a second-order Hurwitz polynomial Q(s) as the denominator of W(s) in equation (4.B.1),

$$Q(s) = s^2 + a_1 s + a_2 \quad ; \quad a_1, a_2 > 0$$

The companion A matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \tag{4.B.6}$$

and by using the procedure of Appendix C the tridiagonal matrix is

$$F = \begin{bmatrix} -a_1 & -\sqrt{a_2} \\ \sqrt{a_2} & 0 \end{bmatrix} \tag{4.B.7}$$

It is easily verified that both A and F have the same characteristic polynomial which is identical to Q(s).

By Theorem I there exists a nonsingular (2·2) transformation matrix T_2 which transforms A into F; i.e.,

$$T_2^{-1} A \, T_2 = F$$

or equivalently

$$A \, T_2 = T_2 \, F \tag{4.B.8}$$

By assuming

$$T_2 = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \tag{4.B.9}$$

and substituting T_2, A, and F into equation (4.B.8),

$$\begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \begin{bmatrix} -a_1 & -\sqrt{a_2} \\ \sqrt{a_2} & 0 \end{bmatrix} \tag{4.B.10}$$

Carry out the multiplication of equation (4.B.10) and equate like entries of the resulting matrices. We now can rewrite these equations as

$$\begin{bmatrix} a_1 & -\sqrt{a_2} & 1 & 0 \\ \sqrt{a_2} & 0 & 0 & -1 \\ a_2 & 0 & 0 & \sqrt{a_2} \\ 0 & a_2 & -\sqrt{a_2} & a_1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = 0 \qquad (4.B.11)$$

This is a homogeneous matrix equation whose coefficient matrix has nullity of 1. One solution of T_2 is

$$T_2 = \begin{bmatrix} 0 & 1/\sqrt{a_2} \\ 1 & 0 \end{bmatrix} \qquad (4.B.12)$$

Since the rank of equation (4.B.11) is three, the solution is not unique. In equation (4.B.12) the value of t_3 is assumed to be unity.

Since an n^{th}-order similarity transformation T_n of equations (4.B.4) is necessary to yield a state model (F,G,H,J),where F is a tridiagonal matrix, and it has not been published in the literature, its general expression is derived for the purpose of simplifying the synthesis procedure. The general expressions for T_n and T_n^{-1} are presented in the following section, and the proof is in included in Appendix D.

3. An n^{th}-Order Similarity Transformation T_n

In this section an n^{th}-order similarity transformation T_n is presented, which transforms the companion state-model realization (A,B,C,D) into a tridiagonal state-model realization (F,G,H,J). Some special properties of T_n are described below.

1) It is found that T_n can be expressed as the product of two matrices,

$$T_n = P_n \cdot K_n^{-1} \tag{4.B.13}$$

where P_n is a quasi-triangular matrix as shown in equation (4.B.16a) and K_n is a diagonal matrix as shown in equation (4.B.15). The k_i's are obtained from the first column of the modified Routh array.

2) The inverse of T_n, T_n^{-1}, can also be expressed as a product of two matrices,

$$T_n^{-1} = K_n \cdot Q_n \tag{4.B.14}$$

where Q_n is a quasi-triangular matrix as shown in equation (4.B.17a) and K_n is the same matrix of equation (4.B.15). It is noted that $Q_n = P_n^{-1}$.

3) T_n is increased in the direction as shown by the dashed lines for the second and the third order cases. Note that each higher-order term is expressed in terms

of lower-order terms and k_i's. For any order n, T_n can be computed from these sets of equations when the k_i's are known.

4) It is seen that in both P_n and Q_n each nonzero entry is separated by a zero both column and row-wise. Also k_1 does not occur in the transformation matrices. A numerical example is given in the next section. It illustrates the procedure of computing a similarity transformation from the equations of this section.

$$K_n = \begin{bmatrix} 1 & & & & \\ & k_2 & & & \bigcirc \\ & & k_2k_3 & & \\ & \bigcirc & & \ddots & \\ & & & & k_2k_3k_4\cdots k_n \end{bmatrix} \tag{4.B.15}$$

$$
P_n = \begin{bmatrix}
 & & & & & & 1 \\
 & & & & & 1 & 0 \\
 & \bigcirc & & & 1 & 0 & -k_n^2 \\
 & & & \cdot & \cdot & \cdot & \cdot \\
 & & & 1 & \cdot & & \cdot \\
 & & 1 & 0 & \cdot & & \cdot \\
 & 1 & 0 & -(k_4^2+\dots+k_n^2) & & & p_{n-1,n} \\
1 & 0 & -(k_3^2+\dots+k_n^2) & 0 & \cdots & p_{n,n-1} & p_{nn}
\end{bmatrix}
$$

(4.B.16a)

where

$$p_{ij} = 0 \; ; \text{ if } (i+j) < (n+1), \text{ or } d \equiv (i+j)-(n+1) \text{ is odd,}$$

$$= 1 \; ; \text{ if } (i+j) = (n+1),$$

$$= (-1)^{d/2} \sum_{m=0}^{n-j} k_{j+m}^2 \left| p_{i-1-m,\,j-1+m} \right| \; ; \text{ otherwise.}$$

(4.B.16b)

$$
Q_n = \begin{bmatrix}
q_{11} & q_{12} & \cdot & \cdot & \cdot & 0 & (k_3^2+\dots+k_n^2) & 0 & 1 \\
q_{21} & & & & & (k_4^2+\dots+k_n^2) & 0 & 1 & \\
\cdot & & & & \cdot & 0 & 1 & & \\
\cdot & & & & \cdot & 1 & & & \\
\cdot & & \cdot & \cdot & \cdot & & & & \\
 & \cdot & \cdot & \cdot & & & & & \\
k_n^2 & 0 & 1 & & & & & & \\
0 & 1 & & & & \bigcirc & & & \\
1 & & & & & & & &
\end{bmatrix}
\tag{4.B.17a}
$$

where

$$
\begin{aligned}
q_{ij} &= 0 \ ; \text{ if } (i+j) > (n+1), \text{ or } d \equiv (n+1)-(i+j) \text{ is odd}, \\
&= 1 \ ; \text{ if } (i+j) = (n+1), \\
&= \sum_{m=0}^{j-1} k_{i+2+m}^2 \, q_{i+2+m,\,j-m} \ ; \text{ otherwise}.
\end{aligned}
\tag{4.B.17b}
$$

4. A Transformation Matrix Example

Given a transfer function $W(s)$

$$W(s) = \frac{2s^3 + 5s^2 + 3s + 9}{s^4 + 2s^3 + 10s^2 + 10s + 17}$$

The companion A matrix and the tridiagonal matrix of $W(s)$ are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -17 & -10 & -10 & -2 \end{bmatrix}$$

and

$$F = \begin{bmatrix} -k_1 & -k_2 & 0 & 0 \\ k_2 & 0 & -k_3 & 0 \\ 0 & k_3 & 0 & -k_4 \\ 0 & 0 & k_4 & 0 \end{bmatrix}$$

where $k_1 = 2$, $k_2 = \sqrt{5}$, $k_3 = \sqrt{8/5}$, and $k_4 = \sqrt{17/5}$.
From equation (4.B.15) the diagonal matrix K_4 is

$$K_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & \sqrt{8} & 0 \\ 0 & 0 & 0 & \sqrt{136/5} \end{bmatrix}$$

From equations (4.B.16a) and (4.B.16b) the matrix P_4 is

$$P_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -17/5 \\ 1 & 0 & -5 & 0 \end{bmatrix}$$

Then from equation (4.B.13) the similarity transformation is

$$T_4 = \begin{bmatrix} 0 & 0 & 0 & \sqrt{5/136} \\ 0 & 0 & 1/\sqrt{8} & 0 \\ 0 & 1/\sqrt{5} & 0 & -\sqrt{17/40} \\ 1 & 0 & -5/\sqrt{8} & 0 \end{bmatrix} \qquad (4.B.18)$$

The inverse of T_4 can be computed from equation (4.B.18) or from equation (4.B.14); it is

$$T_4^{-1} = \begin{bmatrix} 0 & 5 & 0 & 1 \\ 17/\sqrt{5} & 0 & \sqrt{5} & 0 \\ 0 & \sqrt{8} & 0 & 0 \\ \sqrt{136/5} & 0 & 0 & 0 \end{bmatrix}$$

It is verified that $T_4^{-1} A T_4 = F$. This transformation matrix T_4 will be used for the transfer-function synthesis in the following section.

C. SYNTHESIS OF TRANSFER FUNCTIONS

1. Outline of the Synthesis Procedure

The general procedure for the synthesis of a proper transfer function W(s) as given in equation (3.A.2) is as follows:

Step 1) Obtain a companion state model (A,B,C,D) for W(s) as shown in equations (4.B.3).

Step 2) Obtain the k_i's from the modified Routh array.

Step 3) Compute the similarity transformation T_n from equation (4.B.13).

Step 4) Find matrices (G,H,J) from the equations in Theorem I:

$$G = T^{-1} B \tag{4.C.1}$$

$$H = C\ T \tag{4.C.2}$$

$$J = D \tag{4.C.3}$$

Step 5) Realize the state model (F,G,H,J).

The matrix F is tridiagonal with the form shown in equation (3.C.7) and is realized by a RC-gyrator ladder network as illustrated in Section 3.C.4. The matrices (G,H,J) relate the input source to the ladder network and the output from the network. They are discussed in detail for different kinds of transfer functions in the following sections.

2. Synthesis of the Open-Circuit Transfer Impedance Function $Z_{21}(s)$

The transfer-impedance function $Z_{21}(s)$ relates the input current $I_{in}(s)$ to the output voltage $V_o(s)$ by

$$V_o(s) = Z_{21}(s)\ I_{in}(s) \tag{4.C.4}$$

where $Z_{21}(s)$ has the form given in equation (3.A.2). At step 4 in the procedure outlined in Section 4.C.1, a state model of $W(s)$ is obtained, which is

$$\dot{x} = \mathbf{F}\ x + G\ u \tag{4.C.5}$$

$$y = H\ x + J\ u \tag{4.C.6}$$

The matrix F is in tridiagonal form. The matrix G, as derived from equation (4.C.1), is

$$G = T_n^{-1} B$$
$$= \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{4.C.7}$$

Thus, the state equations for Z_{21} have the following form:

$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ V_{C2} \\ \cdot \\ \cdot \\ \cdot \\ V_{Cn} \end{bmatrix} = \begin{bmatrix} -k_1 & -k_2 & & & \bigcirc \\ k_2 & 0 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -k_n \\ \bigcirc & & & k_n & 0 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ \cdot \\ \cdot \\ \cdot \\ V_{Cn} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} I_{in} \tag{4.C.8}$$

From the realization procedure of Section 3.C.4 and equation (4.C.8), the current source I_{in} is shown to be at the input terminals of the ladder network and in parallel with the resistor R_1 as shown in Figure 4-1.

This can be proved by taking the KCL equation at node 1 of Figure 4-1,

$$\dot{V}_{C1} + \frac{1}{R_1} V_{C1} + g_{12}V_{C2} = I_{in} \qquad (4.C.9)$$

where $R_1 = 1/k_1$ and $g_{12} = k_2$. Equation (4.C.9) is the same form as the first equation of (4.C.8). Therefore, Figure 4-1 is the realized network of the state equation (4.C.8).

Now consider the output equation. The matrix H, as derived from equation (4.C.2), is

$$H = C\ T$$

$$= \begin{bmatrix} h_n & h_{n-1} & \cdots & h_1 \end{bmatrix} \qquad (4.C.10)$$

where h_i's are scalar constants. For a proper scalar transfer function as defined in equation (3.A.2), matrix D is zero. From equation (4.C.3)

$$J = D = 0$$

Therefore, the output equation for $Z_{21}(s)$ is

$$V_o = \begin{bmatrix} h_n & h_{n-1} \cdots & h_1 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ \cdot \\ \cdot \\ \cdot \\ V_{Cn} \end{bmatrix} \qquad (4.C.11)$$

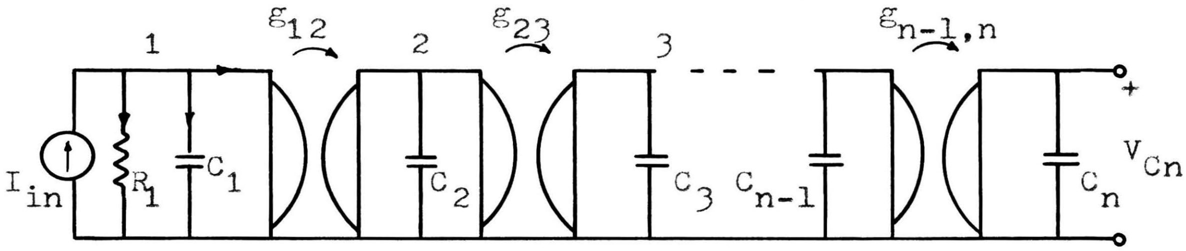


Figure 4-1. The synthesized network of the state equation (4.C.8)

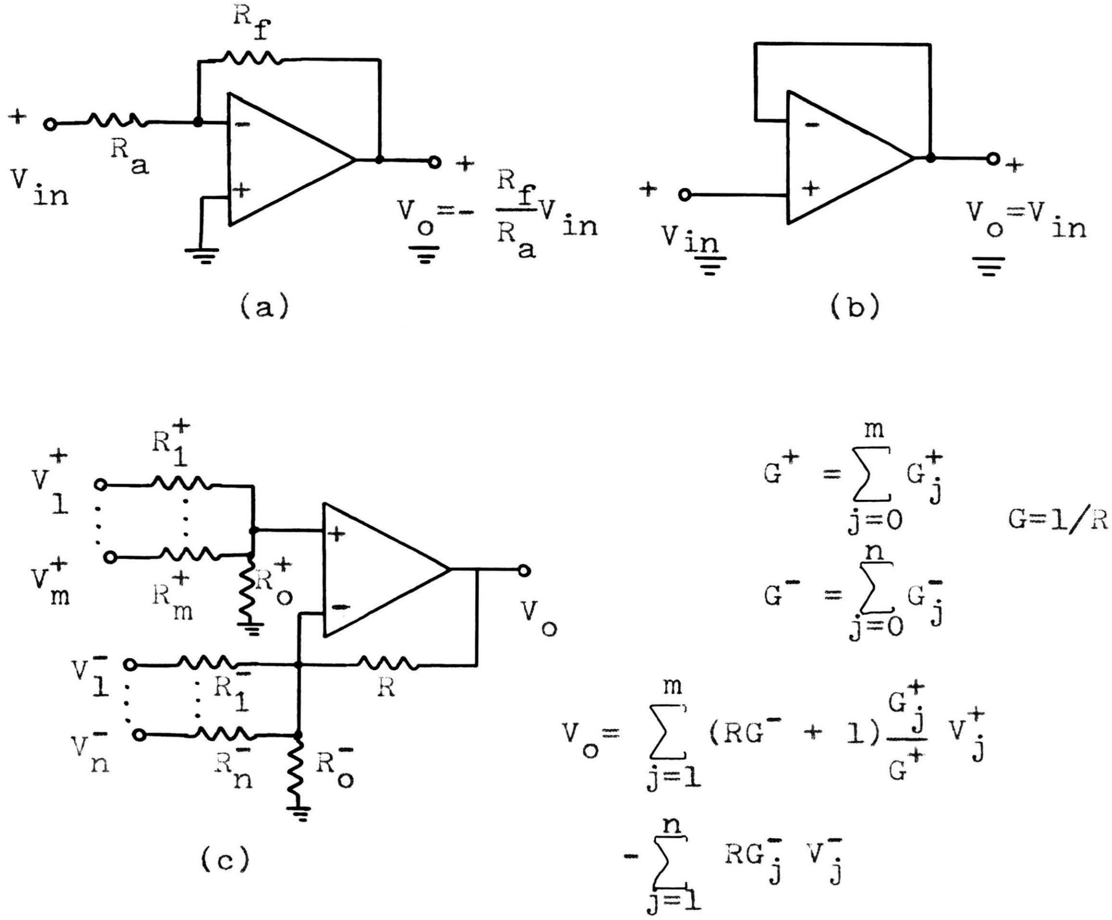


Figure 4-2. (a) A scalar circuit
(b) a buffer circuit and
(c) a summing circuit

<u>Special Case</u>: $H = \begin{bmatrix} 0 & 0 & \cdots & 0 & h_1 \end{bmatrix}$

Presentation of the special case, for which h_1 is the only nonzero entry of H, will help in understanding the general case presented next. Consider a transfer function $Z_{21}(s)$ that has only one nonzero coefficient term in its numerator,

$$Z_{21}(s) = \frac{c_n}{s^n + a_1 s^{n-1} + \ldots + a_n} \tag{4.C.12}$$

The matrix C is

$$C = \begin{bmatrix} c_n & 0 & \cdots & 0 \end{bmatrix}$$

and the matrix H as derived from equation (4.C.2) has the form

$$H = \begin{bmatrix} 0 & \cdots & 0 & h_1 \end{bmatrix}$$

Then the output is

$$V_o = h_1 V_{Cn}$$

where V_{Cn} is the voltage across the capacitor C_n. V_o can be obtained through a scalar circuit (an inverter) which is shown in Figure 4-2a with the resistor ratio $R_f/R_a = h_1$. It is assumed that the input impedance Z_{in} of the scalar is much higher than the impedance across the capacitor Z_{Cn}, such that the loading effect is negligible.

Note that there are other special cases with only one nonzero h_i.

In general, the h_i's are not zeros, the output voltage V_o can be obtained by using a summing circuit (summer) as shown in Figure 4-2c (9). A buffer circuit as shown in Figure 4-2b may be used to reduce the loading effect. The following example is given for illustration.

Example. Realize the following open-circuit transfer impedance function (31);

$$Z_{21}(s) = \frac{2s^3 + 5s^2 + 3s + 9}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (4.C.14)$$

Follow the procedure outlined in Section 4.C.1.

Step 1) The companion state model (A,B,C,D) for $Z_{21}(s)$ is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -17 & -10 & -10 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 9 & 3 & 5 & 2 \end{bmatrix}$$

$$D = 0$$

Step 2) Calculate the modified Routh array to obtain the k_i's as illustrated in Appendix C.

$$k_1 = 2$$
$$k_2 = \sqrt{5}$$
$$k_3 = \sqrt{8/5}$$
$$k_4 = \sqrt{17/5}$$

Step 3) The similarity transformation T_4 and T_4^{-1} are already computed in the example of Section 4.B.4.

$$T_4 = \begin{bmatrix} 0 & 0 & 0 & \sqrt{5/136} \\ 0 & 0 & 1/\sqrt{8} & 0 \\ 0 & 1/\sqrt{5} & 0 & -\sqrt{17/40} \\ 1 & 0 & -5/\sqrt{8} & 0 \end{bmatrix}$$

$$T_4^{-1} = \begin{bmatrix} 0 & 5 & 0 & 1 \\ 17/5 & 0 & \sqrt{5} & 0 \\ 0 & \sqrt{8} & 0 & 0 \\ \sqrt{136/5} & 0 & 0 & 0 \end{bmatrix}$$

Step 4) From equations (4.C.1), (4.C.2), and (4.C.3), the matrices (G,H,J) are obtained, and the state model is shown below.

$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ V_{C2} \\ V_{C3} \\ V_{C4} \end{bmatrix} = \begin{bmatrix} -2 & -\sqrt{5} & 0 & 0 \\ \sqrt{5} & 0 & -\sqrt{8/5} & 0 \\ 0 & \sqrt{8/5} & 0 & -\sqrt{17/5} \\ 0 & 0 & \sqrt{17/5} & 0 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ V_{C3} \\ V_{C4} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} I_{in}$$

$$V_o = \begin{bmatrix} 2 & \sqrt{5} & -7/\sqrt{8} & -\sqrt{40/17} \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ V_{C3} \\ V_{C4} \end{bmatrix}$$

This state model is realized by the circuit shown in Figure 4-3, where the output is obtained through a summer.

The state-variable method (9) realizes the state model (A,B,C,D), which is obtained by inspection as shown in Step 1, with four integrators and two summers.

The passive synthesis procedure (31) requires one transformer and four ladder networks in parallel; each ladder network has two resistors, two capacitors and two inductors.

In Figure 4-3 four buffers may be required to avoid the loading from the summer.

$R_1^- = (\sqrt{8}/7)\ R$

$R_2^- = (\sqrt{17/40})R$

$C_1 = C_2 = C_3 = C_4 = 1$

$R_1^+ = 0.88\ R$

$R_2^+ = 0.78\ R$

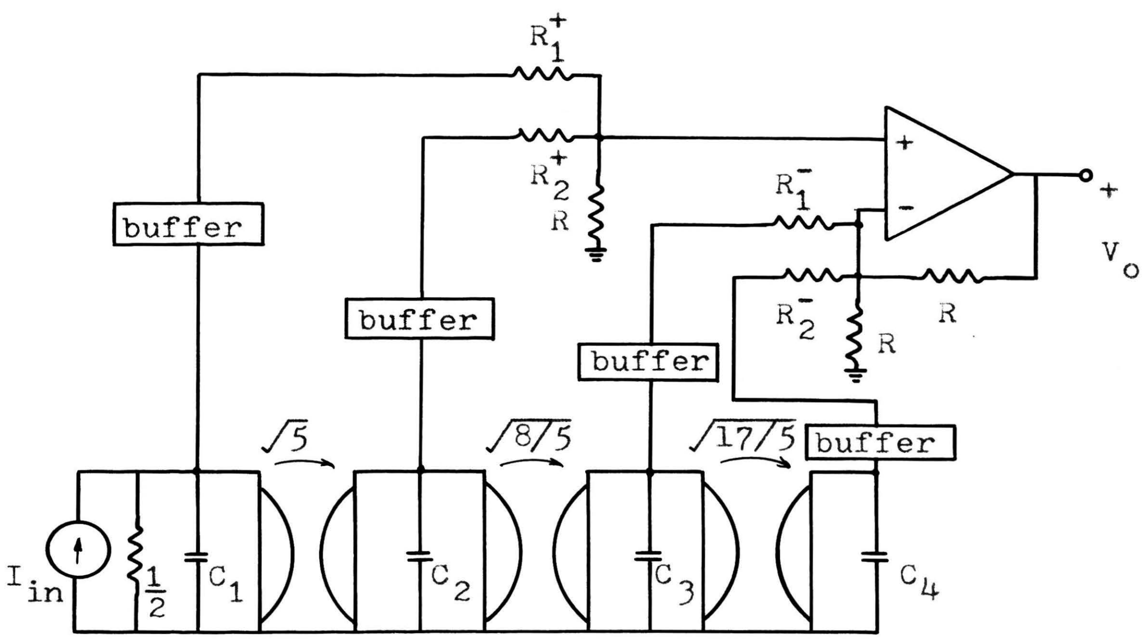


Figure 4-3. The synthesized network of equation (4.C.14)

3. Synthesis of the Voltage Transfer Function T(s)

The voltage transfer function T(s) relates the input voltage V_{in} and the output voltage V_o by the relation

$$V_o(s) = T(s)\, V_{in}(s)$$

and has the form given in equation (3.A.2). Following the procedure outlined in Section 4.C.1, the state equation for T(s) is found to be

$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ V_{C2} \\ \vdots \\ V_{C_{n-1}} \\ V_{Cn} \end{bmatrix} = \begin{bmatrix} -k_1 & -k_2 & & & \bigcirc \\ k_2 & 0 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ \bigcirc & & k_{n-1} & 0 & -k_n \\ & & & k_n & 0 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ \vdots \\ V_{C_{n-1}} \\ V_{Cn} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} V_{in} \tag{4.C.15}$$

Equation (4.C.15) is similar to equation (4.C.8), except that a voltage source is the forcing function in this case.

The output equation has the form as for $Z_{21}(s)$,

$$V_o = \begin{bmatrix} h_n & h_{n-1} & \cdots & h_1 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ \vdots \\ V_{Cn} \end{bmatrix} \tag{4.C.16}$$

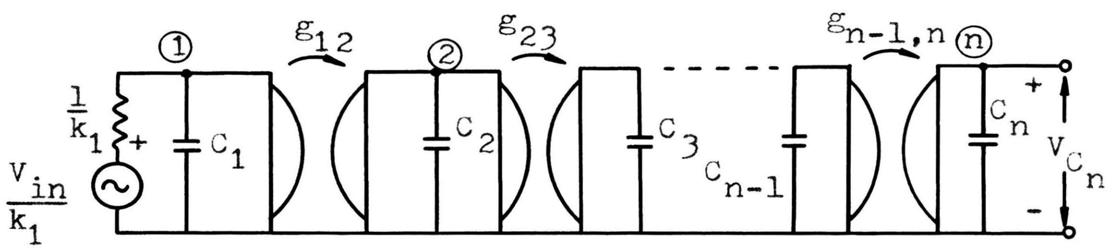


Figure 4-4. The synthesized network of the state equation (4.C.15)

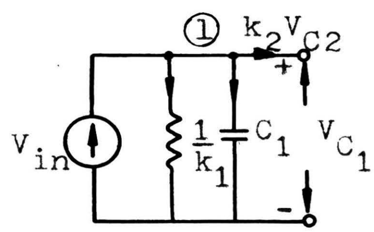


Figure 4-5. The equivalent network at node 1 of Figure 4-4

Equation (4.C.15) is realized by the ladder network as shown in Figure 4-4. Note that the voltage source has a value of V_{in}/k_1 and is in series with a resistor whose value is $1/k_1$. This result can be proved by first taking the KCL equation at node 1 in the equivalent circuit of Figure 4-5, which is

$$\dot{V}_{C1} + k_1 V_{C1} + k_2 V_{C2} = V_{in} \tag{4.C.17}$$

and seeing that it is the same as the first equation of (4.C.15). The output equation is similarly realized as for the case of $Z_{21}(s)$. A second-order filter is realized in the following example for illustrative purposes.

<u>Example</u> Realize the following Butterworth function

$$T(s) = \frac{1/2}{s^2 + \sqrt{2}\, s + 1}$$

Step 1) The state model (A,B,C,D) is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1/2 & 0 \end{bmatrix}$$

$$D = 0$$

Step 2) Since A is in tridiagonal form, the similarity transformation T is not necessary in this case.

Step 5) The state equation and the output equation for T(s) are

$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} V_{C1} \\ V_{C2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{in}$$

$$V_o = 1/2\ V_{C1}$$

This state model is realized with a RC-gyrator ladder network as shown in Figure 4-6, where an inverter is used to obtain the output with a scale factor $R_f/R_1 = 1/2$.

Many authors have presented the synthesis of voltage transfer functions. It is interesting to compare the synthesized networks by different procedures for the same transfer function in the above example.

The network synthesized by the state-variable method (9) uses two integrators and one summer. The passive synthesis procedure (24) requires two capacitors, three gyrators and one resistor.

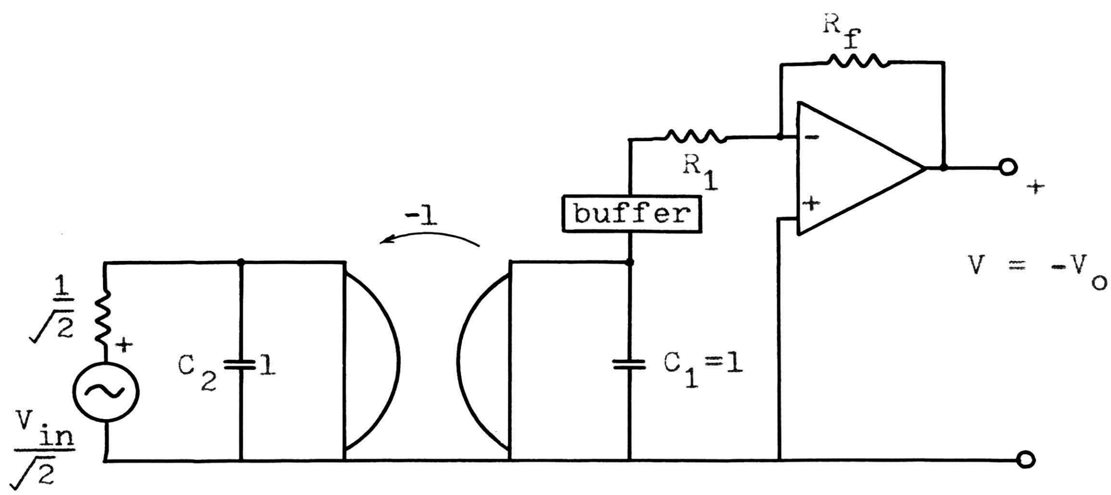


Figure 4-6. The synthesized network of Example (4.C.3)

4. Synthesis of the Short-Circuit Transfer Admittance Function $Y_{21}(s)$

The short-circuit transfer admittance function $Y_{21}(s)$ relates the input voltage $V_{in}(s)$ to the output current $I_o(s)$ by the following equation:

$$I_o(s) = Y_{21}(s)\ V_{in}(s)$$

where $Y_{21}(s)$ has the same form as given in equation (3.A.2). Following the procedure in Section 4.C.1, the state equation is obtained and has the same form as equation (4.C.15). The output equation for $Y_{21}(s)$ is

$$I_o = \begin{bmatrix} h_n & h_{n-1} \cdots & h_1 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ \cdot \\ \cdot \\ \cdot \\ V_{Cn} \end{bmatrix} \qquad (4.C.18)$$

The output current can be obtained through a summer and a high-value resistor or a voltage-to-current converter (VIC). See the example below. Several operational amplifier VIC circuits discussed in the reference (27) may be used.

<u>Example</u>. Synthesize the following short-circuit transfer admittance function,

$$Y_{21}(s) = \frac{-2s^2 - 12}{s^3 + 5s^2 + 17s + 25}$$

Step 1) The matrices (A,B,C,D) for $Y_{21}(s)$ are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -25 & -17 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -12 & -2 & 0 \end{bmatrix}$$

$$D = 0$$

Step2) The k_i's from the modified Routh array are:

$$k_1 = 5, \; k_2 = \sqrt{12}, \text{ and } k_3 = \sqrt{5}$$

Step 3) The similarity transformation matrix T_3 as calculated from equation (4.B.13) is

$$T_3 = \begin{bmatrix} 0 & 0 & 1/\sqrt{5} \\ 0 & 1/\sqrt{12} & 0 \\ 1 & 0 & -\sqrt{5} \end{bmatrix}$$

Step 4) The matrix H is

$$H = CT$$

$$= \begin{bmatrix} 0 & -1/\sqrt{3} & -12/\sqrt{5} \end{bmatrix}$$

Then the state model for $Y_{21}(s)$ is

$$\frac{d}{dt}\begin{bmatrix} V_{C1} \\ V_{C2} \\ V_{C3} \end{bmatrix} = \begin{bmatrix} -5 & -\sqrt{12} & 0 \\ \sqrt{12} & 0 & -\sqrt{5} \\ 0 & \sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ V_{C3} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} V_{in}$$

$$I_o = \begin{bmatrix} 0 & -1/\sqrt{3} & -12/\sqrt{5} \end{bmatrix} \begin{bmatrix} V_{C1} \\ V_{C2} \\ V_{C3} \end{bmatrix}$$

Step 5) The synthesized network of this state model is shown in Figure 4-7. It may require two buffers to eliminate the loading from the summer.

The state-variable method (9) requires three integrators, two summers and one VIC.

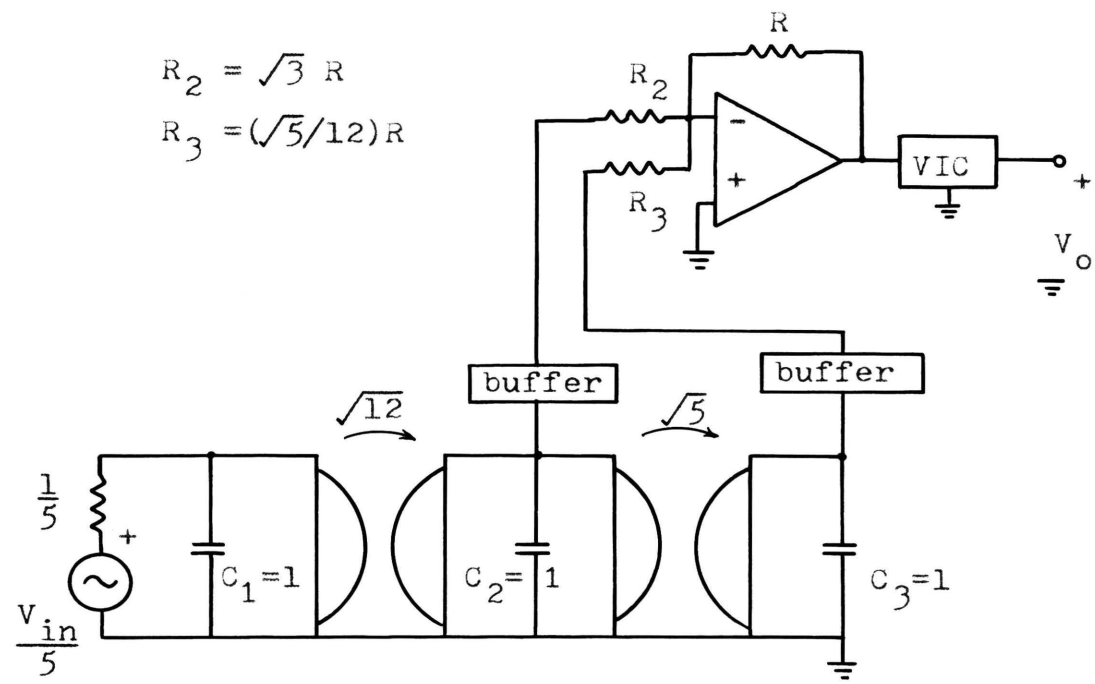


Figure 4-7. The synthesized network of Example (4.C.4)

V. RLC SYMMETRICAL LATTICE NETWORK SYNTHESIS

A. INTRODUCTION

The last two chapters presents a procedure for transfer-function synthesis using a modern system theory approach. The synthesized network contains a minimal number of capacitors which is equal to the denominator degree of the transfer function. The given transfer function W(s) is assumed to have no common factors in its numerator and denominator. Also this assumption is a necessary and sufficient condition for the matrices (A,B,C,D) to be a minimal realization of W(s) as stated in Section 4.B.1.

This chapter presents a procedure for realizing an A matrix with RLC symmetrical lattice networks. It is observed that the number of reactive elements of a symmetrical lattice network is greater than the denominator degree of its transfer function. Therefore, it is not a minimal network, and Theorem II proves that there ia cancellation of a common factor in the network functions. Also Theorem III proves that the eigenvalues of the symmetrical lattice A matrix are the same as the poles and zeros of its driving-point function after cancellation.

A given A matrix is realized with a RC, RL, LC, or RLC symmetrical lattice network, depending upon the locations of its eigenvalues. See flow chart in Figure 5-1.

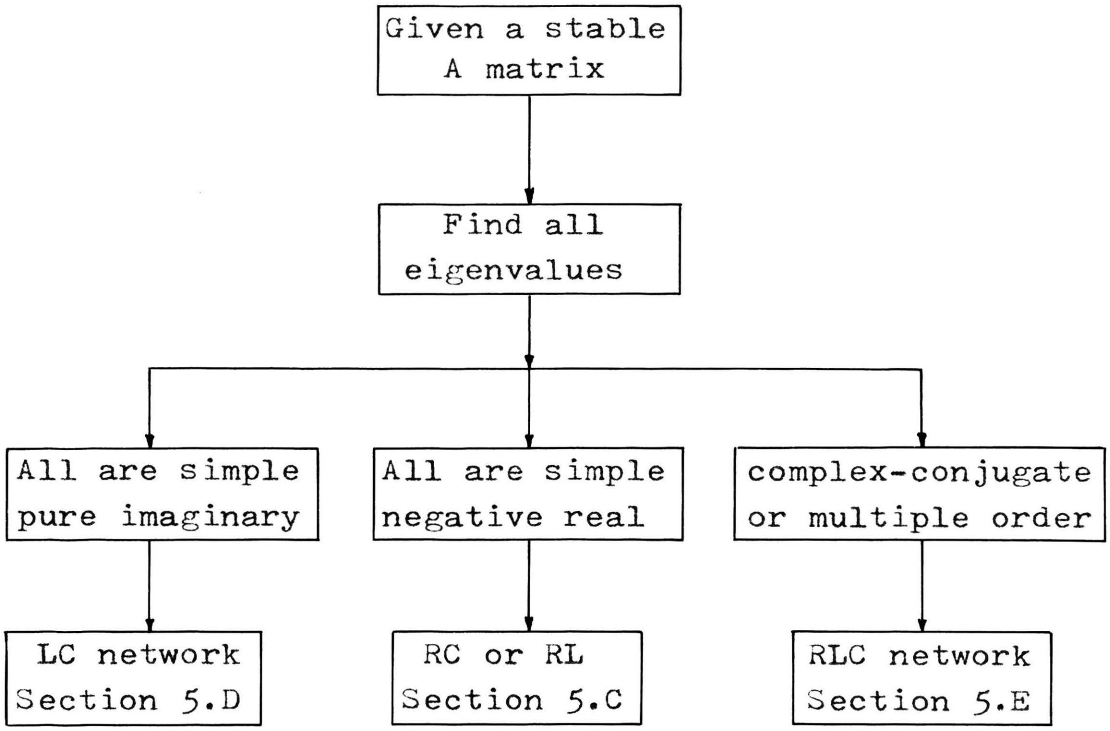


Figure 5-1. The flow chart for the procedure of A-matrix synthesis with symmetrical lattice networks

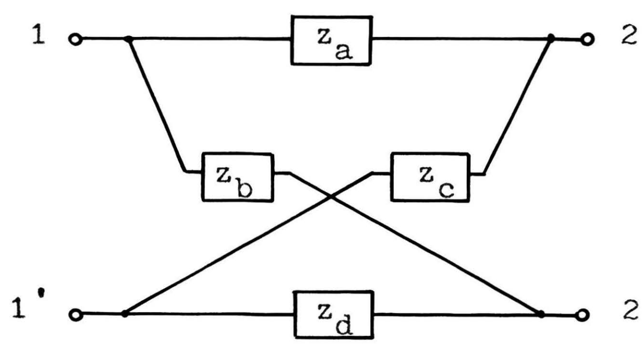


Figure 5-2. A RLC lattice network

B. ANALYSIS

When comparing the eigenvalues of the A matrix with the poles and zeros of the network functions (driving-point and transfer functions) of a symmetrical lattice network, the well-known property of Theorem II is observed and the new property of Theorem III is found. Both theorems and their proofs are given below.

Theorem II: For a symmetrical lattice network each network function has a common factor in its numerator and denominator.

The proof is as follows. Consider a nonsymmetrical lattice network S as shown in Figure 5-2. The open-circuit parameters are

$$z_{11} = \frac{(z_a + z_c)(z_b + z_d)}{(z_a + z_b + z_c + z_d)} \qquad (5.B.1)$$

and

$$z_{12} = \frac{(z_b + z_d)z_c - (z_a + z_c)z_d}{(z_a + z_b + z_c + z_d)} \qquad (5.B.2)$$

If S is symmetric, $z_c = z_b$ and $z_d = z_a$; then z_{11} and z_{12} become

$$z_{11} = \frac{(z_a + z_b)^2}{2(z_a + z_b)} \qquad (5.B.3)$$

and

$$z_{12} = \frac{(z_b - z_a)(z_a + z_b)}{2(z_a + z_b)} \qquad (5.B.4)$$

It is seen that a common factor exists when S becomes symmetrical.

When the common factor $(z_a + z_b)$ in z_{11} and z_{12} is removed, equations (5.B.3) and (5.B.4) become

$$z_{11} = \frac{1}{2}(z_a + z_b) \tag{5.B.5}$$

and

$$z_{12} = \frac{1}{2}(z_b - z_a) \tag{5.B.6}$$

which are the same as derived directly from a symmetrical lattice network (2,3).

Next consider the y parameters. From Figure 5-2

$$y_{11} = \frac{(y_a + y_b)(y_c + y_d)}{(y_a + y_b + y_c + y_d)} \tag{5.B.7}$$

and

$$y_{12} = \frac{y_b(y_a + y_c) - y_a(y_b + y_d)}{(y_a + y_b + y_c + y_d)} \tag{5.B.8}$$

where $y_a = 1/z_a$, $y_b = 1/z_b$, $y_c = 1/z_c$, and $y_d = 1/z_d$.

If S is symmetric, $y_c = y_b$ and $y_d = y_a$; then

$$y_{11} = \frac{(y_a + y_b)^2}{2(y_a + y_b)} \tag{5.B.9}$$

and

$$y_{12} = \frac{(y_b - y_a)(y_a + y_b)}{2(y_a + y_b)} \tag{5.B.10}$$

It is seen that there is a common factor in the numerators and denominators of y_{11} and y_{12}. When the common factor is removed, (2,3),

$$y_{11} = \frac{1}{2}(y_a + y_b) \qquad (5.B.11)$$

and

$$y_{12} = \frac{1}{2}(y_b - y_a) \qquad (5.B.12)$$

The proofs for z_{22}, z_{21}, y_{22}, and y_{21} are similar and are omitted. Theorem II can also be proved by using topological formulas for network functions (10,34).

The following theorem shows the relation between the eigenvalues of the A matrix and the critical frequencies of the driving-point functions.

Theorem III: The eigenvalues of the A matrix of a symmetrical lattice network are the same as the zeros and the poles of the driving-point function with the common factor removed.

The proof is as follows. For a symmetrical lattice network some poles have been cancelled by the zeros as shown in Theorem II. In the proof of this theorem, we only need to show that the poles of the network function before cancellation are the same as the poles and zeros of the driving-point function after cancellation. We can find the poles from the denominator of y_{11} (or y_{12}).

In the lattice network S of Figure 5-2, let

$$y_a = N_a/D_a,\ y_b = N_b/D_b,\ y_c = N_c/D_c, \text{ and } y_d = N_d/D_d$$

where the N_i's and D_i's are polynomials of the complex variable s. From equation (5.B.7) the driving-point function of the network S is

$$y_{11} = \frac{\left(\frac{N_a}{D_a} + \frac{N_b}{D_b}\right)\left(\frac{N_c}{D_c} + \frac{N_d}{D_d}\right)}{\left(\frac{N_a}{D_a} + \frac{N_b}{D_b} + \frac{N_c}{D_c} + \frac{N_d}{D_d}\right)}$$

or

$$y_{11} = \frac{(N_aD_b + N_bD_a)(N_cD_d + N_dD_c)}{(N_aD_bD_cD_d + N_bD_aD_cD_d + N_cD_aD_bD_d + N_dD_aD_bD_c)} \tag{5.B.13}$$

If S is symmetrical , $y_c = y_b$ and $y_d = y_a$ which implies that

$$N_c = N_b, \quad D_c = D_b$$

and (5.B,14)

$$N_d = N_a, \quad D_d = D_a$$

then equation (5.B.13) becomes

$$y_{11} = \frac{(N_aD_b + N_bD_a)^2}{2D_aD_b(N_aD_b + N_bD_a)} \tag{5.B.15}$$

After cancelling the common factor,

$$y_{11} = \frac{(N_aD_b + N_bD_a)}{2D_aD_b} \tag{5.B.16}$$

We see that the poles of y_{11} before cancellation are from the denominator polynomial $D_a D_b (N_a D_b + N_b D_a)$ of equation (5.B.15), and are the same as the poles and zeros of y_{11} after cancelling the common factor, equation (5.B.16).

As shown in equation (1.B.2) the denominator of the transfer function before any factors are cancelled is the same as the characteristic polynomial $|sI - A|$. Thus, the eigenvalues of the A matrix, which are same as the poles before cancelling the common factor, are the same as the poles and zeros of y_{11} with the common factor removed. The proof for the open-circuit parameters is similar and is omitted.

Corollary I: A symmetrical lattice network is not a minimal network.

Theorems II and III prove that a symmetrical lattice network is not a minimal network, because some of the poles are cancelled with the zeros in the network functions. Thus, the order of the A matrix or the number of the reactive elements is greater than the degree of the transfer function denominator. Theorem II and Theorem III are illustrated by the following example.

Example For the symmetrical lattice network shown in Figure 5-3a, the z parameters are

$$z_{11} = \frac{R_1}{2} \frac{\left(s^2 + \dfrac{R_1C_1 + R_2C_2 + R_2C_1}{R_1R_2C_1C_2} s + \dfrac{1}{R_1R_2R_3R_4}\right)}{s\left(s + \dfrac{1}{R_2C_2}\right)} \tag{5.B.17}$$

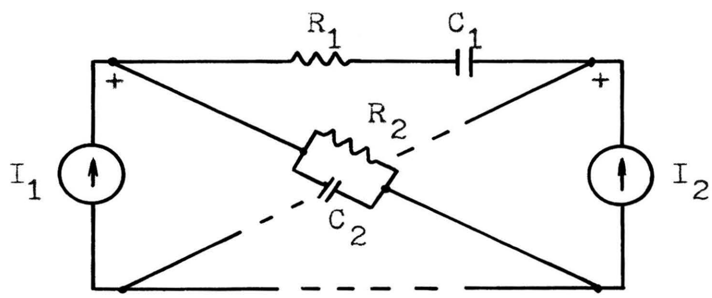
and

$$z_{12} = \frac{-R_1}{2} \frac{\left(s^2 + \dfrac{R_1C_1 + R_2C_2 - R_2C_1}{R_1R_2C_1C_2} s + \dfrac{1}{R_1R_2C_1C_2}\right)}{s\left(s + \dfrac{1}{R_2C_2}\right)} \tag{5.B.18}$$

Note that the common factor in the above two equations has been removed. The denominator is of degree two, but there are four capacitors in the network. The A matrix of this network can be found from a general state-model representation (31),

$$A = \begin{bmatrix} 1/C_1 & 1/C_1 & -1/C_1 & 1/C_1 \\ 1/C_2 & (1+2R_1/R_2)/C_2 & -1/C_2 & -1/C_2 \\ -1/C_2 & -1/C_2 & (1+2R_1/R_2)/C_2 & 1/C_2 \\ -1/C_1 & -1/C_1 & 1/C_1 & 1/C_1 \end{bmatrix} \tag{5.B.19}$$

(a)



(b)

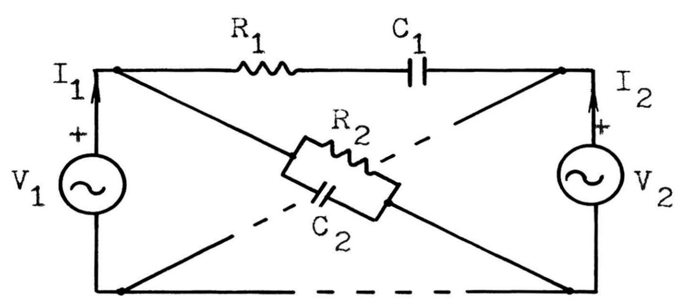


Figure 5-3. A symmetrical lattice
(a) for finding the open-circuit parameters and
(b) for finding the short-circuit parameters

The calculation of the eigenvalues of A is quite tedious. It is found that its characteristic polynomial can be factored as

$$|sI - A| = s\left(s+\frac{1}{R_2C_2}\right)\left(s^2+\frac{R_1C_1+R_2C_2+R_2C_1}{R_1R_2C_1C_2}\,s+\frac{1}{R_1R_2C_1C_2}\right) \tag{5.B.20}$$

By comparing the denominator of equation (5.B.17) or equation (5.B.18) with the characteristic polynomial of A, it can be seen that the quadratic term in equation (5.B.20) does not occur in the denominator of z_{11} or z_{12} because of cancellation as stated by Theorem II. Also this quadratic term is the numerator of z_{11}, and the eigenvalues of A are the same as the poles and zeros of z_{11} as shown by Theorem III.

To demonstrate the y parameters for Figure 5-3b,

$$y_{11} = \frac{C_2}{2}\left(\frac{s^2+\dfrac{R_1C_1+R_2C_2+R_2C_1}{R_1R_2C_1C_2}s+\dfrac{1}{R_1R_2C_1C_2}}{s+\dfrac{1}{R_1C_1}}\right)$$

$$y_{12} = \frac{C_2}{2}\left(\frac{s^2+\dfrac{R_1C_1+R_2C_2-R_2C_1}{R_1R_2C_1C_2}s+\dfrac{1}{R_1R_2C_1C_2}}{s+\dfrac{1}{R_1C_1}}\right)$$

The A matrix is found to be

$$A = \begin{bmatrix} \frac{1}{R_1C_1} & 0 & -\frac{1}{R_1C_1} \\ 0 & \frac{1}{R_1C_1} & \frac{1}{R_1C_1} \\ -\frac{1}{2R_1C_2} & \frac{1}{2R_1C_2} & \frac{1}{R_1C_2}+\frac{1}{R_2C_2} \end{bmatrix}$$

and the characteristic polynomial is

$$|sI-A| = (s+\frac{1}{R_1C_1})(s^2 + \frac{R_1C_1+R_2C_2+R_2C_1}{R_1R_2C_1C_2}s + \frac{1}{R_1R_2C_1C_2})$$

It is seen that the same quadratic factor does not occur in the denominator polynomial but is in the numerator of y_{11} as predicted by Theorems II and III.

The following sections present procedures for A-matrix realization based on Theorem III, on the characteristics of driving-point functions, and on the locations of the eigenvalues of A.

C. SYNTHESIS OF THE A MATRIX WITH RC OR RL SYMMETRICAL LATTICE NETWORKS

1. Characteristics of RC and RL Driving-Point Immittance Functions

Before the synthesis procedure is presented, some important characteristics of driving-point immittance functions (admittance or impedance) of RC and RL networks are listed below for easy reference (2,3).

Let a driving-point immittance function be written as

$$F(s) = k \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = k \frac{P(s)}{Q(s)} \qquad (5.C.1)$$

where k is a scalar constant and $|m-n| = 0$ or 1.

1) P(s) and Q(s) must be strictly Hurwitz.
2) All the roots of P(s) and Q(s) must be simple and alternate on the negative real axis.
3) For RC (or RL) networks, the driving-point impedance function Z_{RC} (Y_{RL}) has a pole closest to the origin and a zero nearest infinity.
   On the contrary, the driving-point admittance function Y_{RC} (Z_{RL}) has a zero nearest the origin and a pole nearest infinity.
   This characteristic is shown in Figure 5-4.

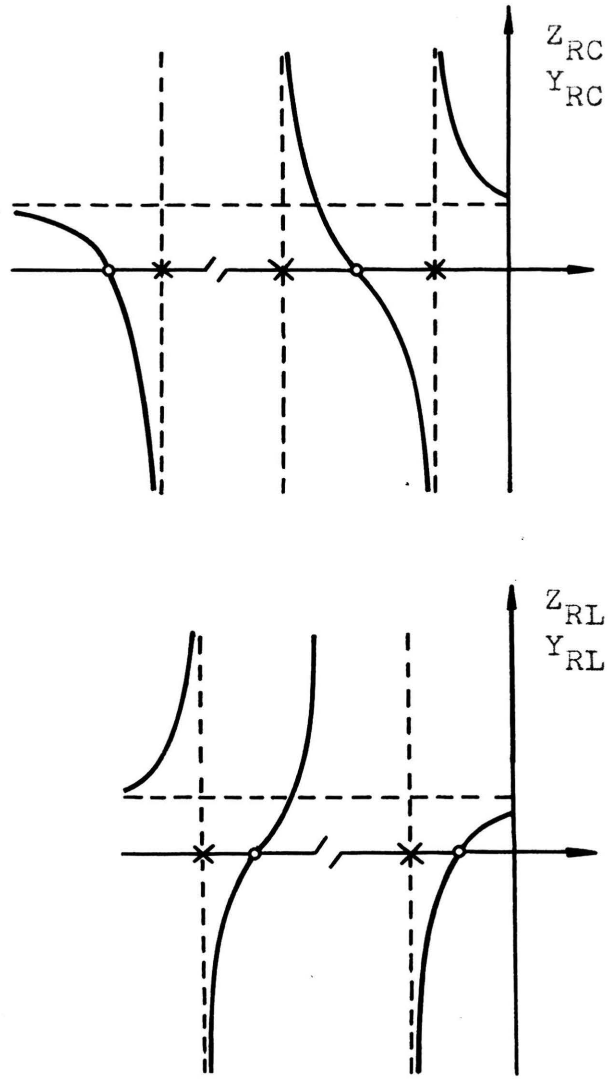


Figure 5-4. Typical plots of RC and RL driving-point functions

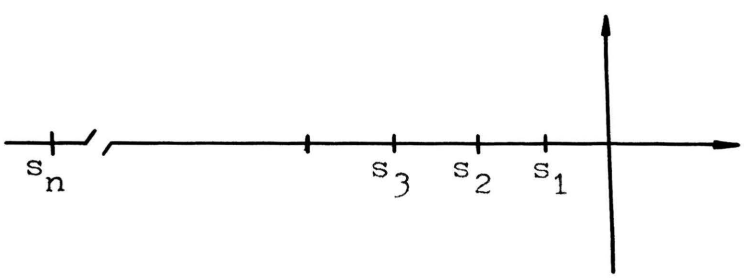


Figure 5-5. Poles and zeros of a RC or RL driving-point function

2. Realization Procedure

When the eigenvalues of a given A matrix are all on the negative real axis and simple, A can be realized with RC or RL symmetrical lattice networks. The following realization procedure is based on the characteristics of RC or RL driving-point functions, and on Theorem III of section 5.B:

Step 1) Find the eigenvalues of A. Check that all eigenvalues are simple and on the negative real axis.

Step 2) Arrange the eigenvalues along the negative real axis in the sequence shown in Figure 5-5.

Step 3) Form a driving-point immittance function according to one of the types as follows:

Type I - RC symmetrical lattice

Case 1) $$Y_{RC} = k\,\frac{(s-s_1)(s-s_3)\cdots}{(s-s_2)(s-s_4)\cdots} \tag{5.C.2}$$

Case 2) $$Z_{RC} = k\,\frac{(s-s_2)(s-s_4)\cdots}{(s-s_1)(s-s_3)\cdots} \tag{5.C.3}$$

Type II - RL symmetrical lattice

Case 1) $$Y_{RL} = k\,\frac{(s-s_2)(s-s_4)\cdots}{(s-s_1)(s-s_3)\cdots} \tag{5.C.4}$$

$$\text{Case 2)} \qquad Z_{RL} = k \frac{(s-s_1)(s-s_3)\cdots}{(s-s_2)(s-s_4)\cdots} \qquad (5.C.5)$$

Step 4) Realize the desired driving-point function formed in Step 3.

In Step 3 the poles and zeros are chosen such that the characteristics of a driving-point immittance function listed in Section 5.C.1 are satisfied.

In Step 4 the driving-point function is expanded by partial fractions and then grouped into two parts, according to equation (5.B.5) or (5.B.11). Since the constant k is arbitrary, the element values of the synthesized network may differ by a multiplicative constant with different k, but the critical frequencies remain the same. It is noted that the synthesized network is not minimal; the number of reactive elements is greater than the number of eigenvalues of A or the denominator degree.

3. An Example

In this example the following A matrix is realized with several symmetrical lattice networks.

$$A = - \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1/2 & 1/2 \\ -1/2 & 1/2 & 4/3 \end{bmatrix}$$

Step 1) The characteristic equation of A is

$$(s + 1/2)(s^2 + 11/6\ s + 1/6) = 0$$

with eigenvalues of

$$s_2 = -1/2$$

$$s_1 = -(11-\sqrt{97})/12$$

$$s_3 = -(11 +\sqrt{97})/12$$

All three eigenvalues are negative and real.

Step 2) The eigenvalues are arranged as shown in Step 1.

Step 3 and Step 4) In this example each case is realized for illustrative purposes.

Type I - RC symmetrical lattice networks

Case 1)

$$Y_{RC}(s) = k\,\frac{s^2 + 11/6\ s + 1/6}{s + 1/2}$$

$$= k\left(s + 1/3 + \frac{s}{s + 1/2}\right) \qquad (5.C.6)$$

Let k = 1/2, the synthesized networks are shown in Figure 5-6 with

a) $y_a = \dfrac{s}{s + 1/2}$; $y_b = s + 1/3$

b) $y_a = \dfrac{s}{s + 1/2} + \dfrac{1}{3}$; $y_b = s$

c) $y_a = \frac{s}{s + 1/2} + s; \quad y_b = 1/3$

Case 2)

$$Z_{RC} = k\frac{s + 1/2}{s^2 + 11/6\ s + 1/6}$$

$$= k\left(\frac{0.25}{s + 0.096} + \frac{0.75}{s + 1.74}\right)$$

By assuming $k = 1/2$ the synthesized network is shown in Figure 5-7 where

$$z_a = \frac{0.25}{s + 0.096}$$

$$z_b = \frac{0.75}{s + 1.74}$$

Type II - RL symmetrical lattice networks

Case 1)

$$Y_{RL} = \frac{s + 1/2}{s^2 + 11/6\ s + 1/6}$$

$$= k\left(\frac{0.25}{s + 0.096} + \frac{0.75}{s + 1.74}\right)$$

Let $k = 1/2$ and choose

$$y_a = \frac{0.25}{s + 0.096} \quad ; \quad y_b = \frac{0.75}{s + 1.74}$$

The synthesized network is shown in Fig. 5-8 case 2)

$$Z_{RL} = k\frac{s^2 + 11/6\ s + 1/6}{s + 1/2}$$

$$= k\left(s + 1/3 + \frac{s}{s + 1/2} \right)$$

The realized networks are shown in Figure 5-9 with $k = 1/2$ and

a) $z_a = \dfrac{s}{s + 1/2}$; $z_b = s + 1/3$

b) $z_a = \dfrac{s}{s + 1/2} + 1/3$; $z_b = s$

c) $z_a = \dfrac{s}{s + 1/2} + s$; $z_b = 1/3$

It can be seen that the networks in Case 2 Type I are the dual networks of Case 1 Type II, and the networks of case 1 Type I are the dual networks of Case 2 Type II.

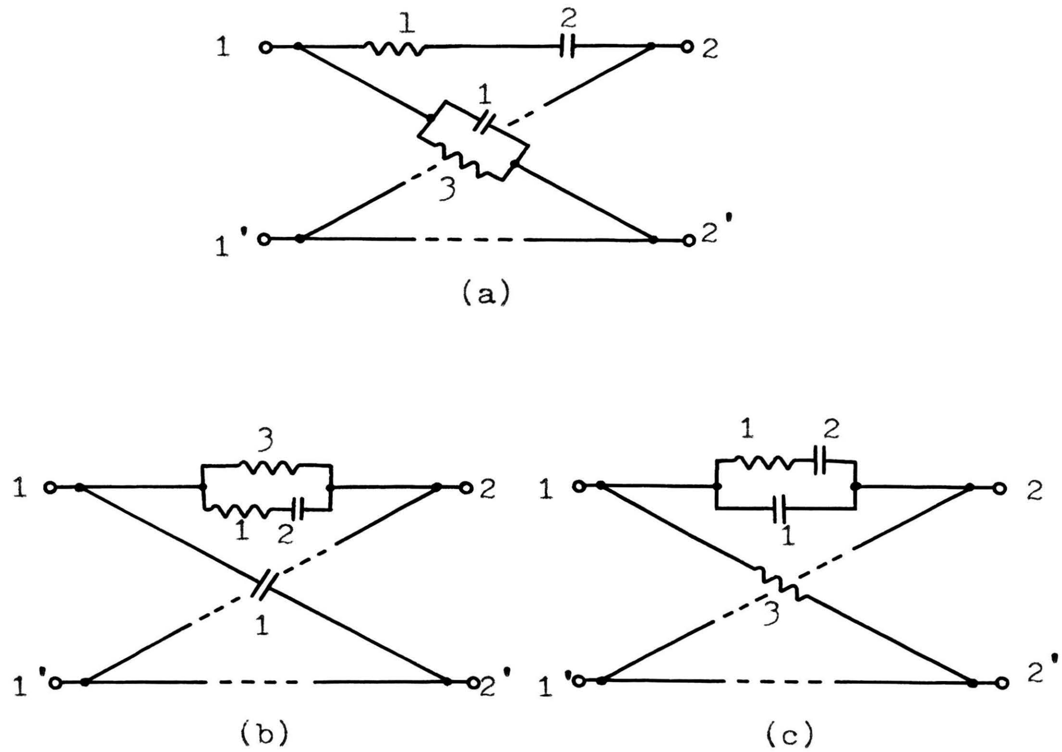


Figure 5-6. The synthesized networks of Case 1, Type I

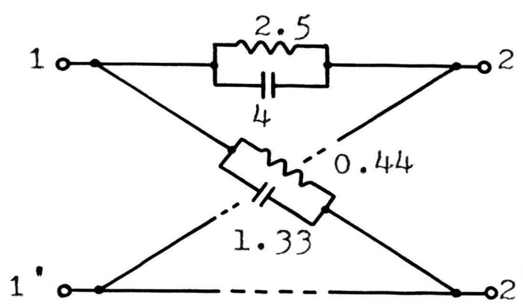


Figure 5-7. The synthesized network of Case 2, Type I

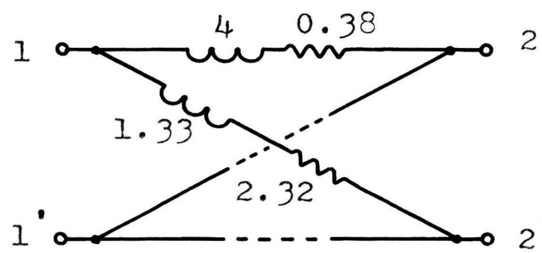
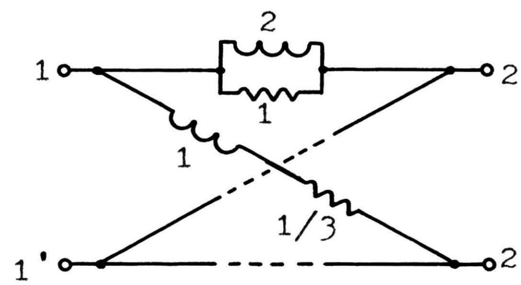
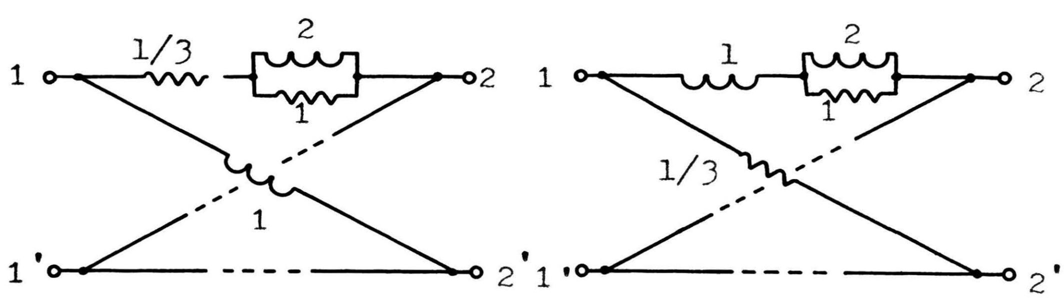


Figure 5-8. The synthesized network of Case 1, Type II



(a)



(b) (c)

Figure 5-9. The synthesized networks of Case 2, Type II

D. SYNTHESIS OF THE A MATRIX WITH LC SYMMETRICAL LATTICE NETWORKS

1. Characteristics of LC Driving-point Immittance Functions

Networks made up of inductors and capacitors are called lossless or reactive networks. It is known that all poles and zeros of a LC driving-point immittance function alternate on the jw-axis (3). The general form of a LC driving-point function (Y_{LC} or Z_{RC}) follows.

$$F(s) = k \frac{(s^2+w_1^2)(s^2+w_3^2)\ldots}{s(s^2+w_2^2)(s^2+w_4^2)\ldots} \qquad (5.D.1)$$

where $0 \leq w_1 < w_2 < w_3 \quad \ldots$

It is observed that F(s) is the ratio of an even polynomial to an odd one, or vice versa.

2. Realization Procedure

When the eigenvalues of an A matrix are all purely imaginary, the A matrix can be realized with a LC symmetrical network by the following procedure. This procedure is based on Theorem III of Section 5.B and the characteristics of LC driving-point functions.

Step 1) Find the eigenvalues of A and see that they are all purely imaginary and simple.

Step 2) Arrange the magnitude of the eigenvalues in sequence, w_1, w_2, w_3, ... etc.

Step 3) Form the driving-point immittance function as shown in equation (5.D.1).

Step 4) Realize the driving-point function formed in Step 3 by the partial-fraction expansion method.

3. An Example

In this example, the following A matrix will be realized with a symmetrical lattice network. Given

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 & 5 & 5 & -5 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 1) The characteristic equation of A is

$$s\,(s^2 + 1)(s^4 + 13s^2 + 10) = 0$$

which has all its roots on the jw-axis.

Step 2) Arrange the magnitude of the roots in sequence as shown in Figure 5-10.

Step 3) Form the driving-point admittance function,

$$Y_{LC}(s) = k\,\frac{s^4 + 13s^2 + 10}{s(s^2 + 1)}$$

Step 4)

$$Y_{LC}(s) = k \left(s + \frac{10}{s} + \frac{2s}{s^2 + 1} \right)$$

Let k = 1/20 and choose

$$y_a = \frac{1}{10} \left(s + 10/s \right)$$

$$y_b = \frac{1}{10} \left(\frac{2s}{s^2 + 1} \right)$$

The synthesized network is shown in Figure 5-11.

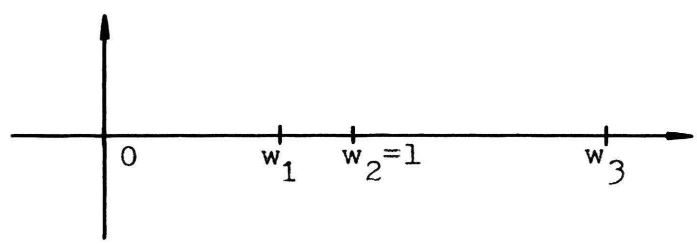


Figure 5-10. The imaginary eigenvalues of Example (5.D.3)

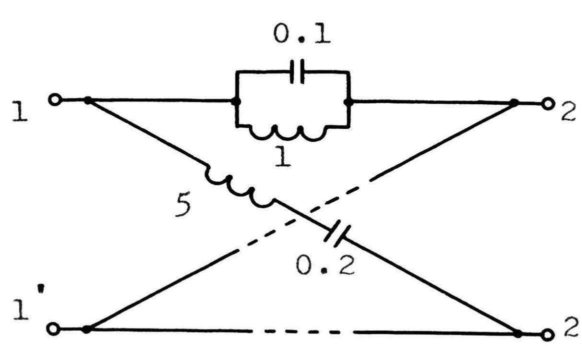


Figure 5-11. The synthesized network of Example (5.D.3)

E. SYNTHESIS OF THE A MATRIX WITH RLC SYMMETRICAL LATTICE NETWORKS

1. Characteristics of RLC Driving-Point Immittance Functions

In this section the procedure for realizing an A matrix with a RLC symmetrical lattice network is presented, whenever the A matrix has complex-conjugate or multiple eigenvlues.

It is known that a driving-point immittance function of a passive network is positive-real. The positive-real characteristic of a rational function F(s) is also a sufficient condition for F(s) to be a driving-point function of a RLC network. This sufficient condition is the foundation of passive network synthesis and was proved by Brune's synthesis procedure (2,3).

Steiglitz and Zemanian (35) have shown, as a sufficient condition, that if the poles and zeros of a rational function F(s) lie on a certain path or region in the left half s-plane (LHP), then F(s) is positive-real. (See Appendix A). The simple alternation of poles and zeros on the negative real axis or jw-axis for the two-element kind networks is a special case of the general results.

2. Realization Procedure

Step 1) Find all the eigenvalues of A and see that they are in LHP and in complex-conjugate pairs.

Step 2) Form a driving-point immittance function F(s)

F(s) is positive-real if complex-conjugate pairs are grouped together as poles and zeros with n-fold alternation on a P-path. (See Appendix A).

Step 3) Realize F(s) with a symmetrical lattice network.

3. An Example

The following A matrix is to be realized with a RLC symmetrical lattice network.

$$A = -\begin{bmatrix} 2 & -0.5 & -0.5 & 0.5 & -0.5 \\ 2 & 0.5 & -0.5 & 1.5 & 1.5 \\ 0 & -0.5 & 0.5 & 0.5 & 0 \\ 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \end{bmatrix}$$

Step 1) The characteristic equation of this A matrix is

$$(s + 2)(s^2 + 10)(s^2 + s + 5) = 0$$

which has all eigenvalues in LHP and in complex-conjugate pairs.

Step 2) Form a driving-point impedance function,

$$Z(s) = k\,\frac{(s^2 + s + 5)}{(s^2 + 10)(\,s + 2)}$$

Step 3) Realize F(s) by partial-fraction expansion,

$$Z(s) = k \left(\frac{0.5}{s + 2} + \frac{0.5\ s}{s^2 + 10} \right)$$

The synthesized network is shown in Figure 5-12.

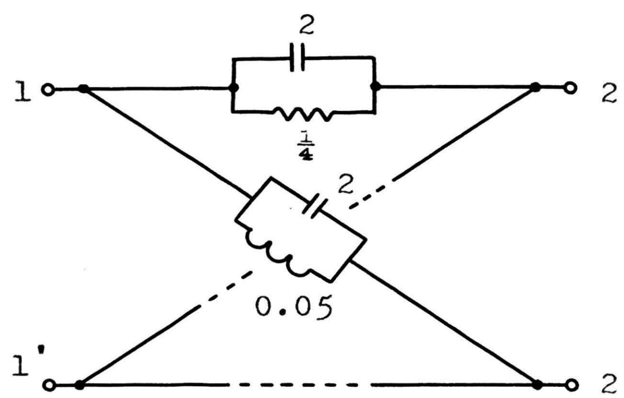


Figure 5-12. The synthesized network of Example (5.E.3)

VI. SUMMARY AND CONCLUSIONS

A. SUMMARY AND CONCLUSIONS

This thesis presents the results of two synthesis procedures. One is the state space approach to the synthesis of transfer functions resulting in RC-gyrator network realizations. The other is a combination of state equation and classical synthesis procedures for realizing the A matrix with symmetrical lattice driving-point immittances. The main results of the state-space synthesis procedure are included in Chapters 3 and 4. In Chapter 3 a simplified A matrix in symbolic form is derived, and the A matrix realization procedure is presented. In Chapter 4 a general expression for the similarity transformation is derived. By this similarity transformation a state model for a given transfer function is obtained with the A matrix in tridiagonal form. The realization procedure for various kinds of transfer functions is presented in separate sections. some properties of this realization procedure follow.

1) It is a unified state-space transfer function synthesis; the procedure is the same regardless of types or characteristics of the transfer functions $Y_{21}(s)$, $Z_{21}(s)$, or $T(s)$. There are no restrictions on the transmission zeros.

2) It is a minimal realization. Capacitors are the only energy storage elements in the synthesized network.

3) The synthesized network contains no inductors or transformers and can be used in integrated circuits. The fact that the gyrators and capacitors share a common ground facilitates fabrication.

4) The realization procedure uses simple algebraic computation. No computer computation is needed even for higher orders.

5) There are restrictions on the topology of the realized network, as mentioned in Section 3.B.3.

6) Since the output is obtained through a summing circuit, loading effect may have to be considered.

Chapter 5 presents a procedure for realizing an A matrix with symmetrical lattice networks. Theorem II proves that each network function of a symmetrical lattice network has a common factor in its numerator and denominator. Theorem III proves that the A matrix eigenvalues are identical to the driving-point function poles and zeros. These results are used in the synthesis of an A matrix. Some properties of this synthesis follow.

1) This A-matrix realization procedure is simple.
2) The synthesized network is a symmetrical lattice in structure and is not minimal.
3) Computer solutions are probably required for solving the A-matrix eigenvalues of higher orders.

B. SUGGESTIONS FOR FURTHER STUDY

An interesting extension of this research would be to find a similarity transformation that transforms an arbitrary A matrix to a companion A matrix or to the desired tridiagonal matrix F, and to determine whether a given state model can be realized by this procedure.

Another idea to investigate is how to realize the A matrix in Schwarz form, which is a tridiagonal matrix similar to the matrix F. A similarity transformation which transforms a companion A matrix into a Schwarz form has been published (33). This transformation is expressed in terms of each column of the Routh array and has a form similar to the transformation matrix T_n of Section 4.B. It is possible that a similar realization procedure can be derived.

Another area of study is how to reduce or eliminate the topological constraints on the network. This will allow the synthesized networks to have more types of structure.

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VITA

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APPENDIX A

SUFFICIENT CONDITIONS ON POLE AND ZERO LOCATIONS FOR RATIONAL POSITIVE-REAL FUNCTIONS

Steiglitz and Zemanian (35) derived the sufficient conditions on pole and zero locations for a rational function F(s) to be positive-real. This appendix gives their important results for easy reference.

P-path: A P-path is generated by moving the complex conjugate pair $(s_p, \bar{s}_p)$ from s = 0 to s = infinity in such a way that $Re(s_p)$ is nonincreasing. Wherever $Re(s_p)$ remains constant on any portion of the P-path, $Im(s_p)$ increases.

Examples of some P-path are shown in Figure A-1. Note that portions of the negative real-axis and LHP vertical lines can be P-paths.

n-fold alternation: A n-fold alternation of poles and zeros on a P-path means that while traversing the P-path in the negative real-axis direction, we encounter first n poles, then n zeros, then n poles, etc., or first n zeros, then n poles, then n zeros, etc.

Examples of 3-fold and 4-fold alternations are shown in Figure A-2. Note that some of the poles and zeros are allowed to be coincident and therefore cancel. In the special case when n=1 and the P-path is the real negative axis, we have the familiar simple alternation of an RL (RC) driving-point function.

<u>Theorem</u>: Let $W(s)=F(s)(s+a)^{\pm 1}$, where the zeros and poles of F(s) have 2-fold alternation on a P-path, and a is real and nonnegative with (-a) no smaller than the right most element of F(s). Also let (-a) be a pole if the starting elements of F(s) are zeros, and a zero if the starting elements of F(s) are poles. Then W(s) is positive-real.

The proof of the theorem and for the cases when n is greater than three can be found in (35).

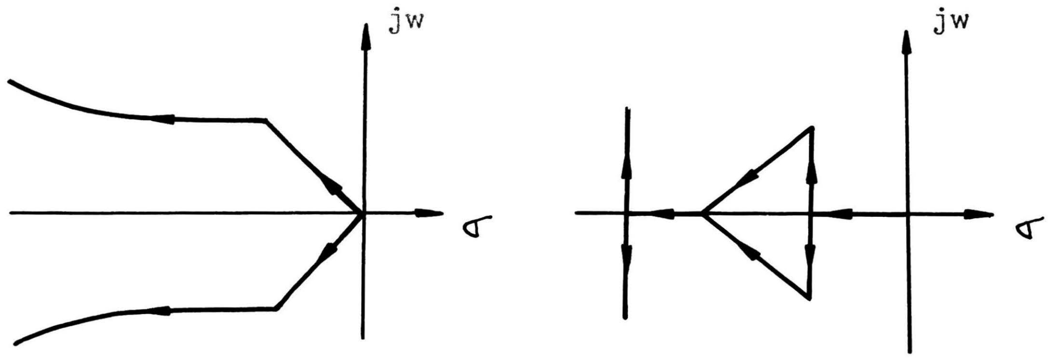


Figure A-1. Examples of some P-paths

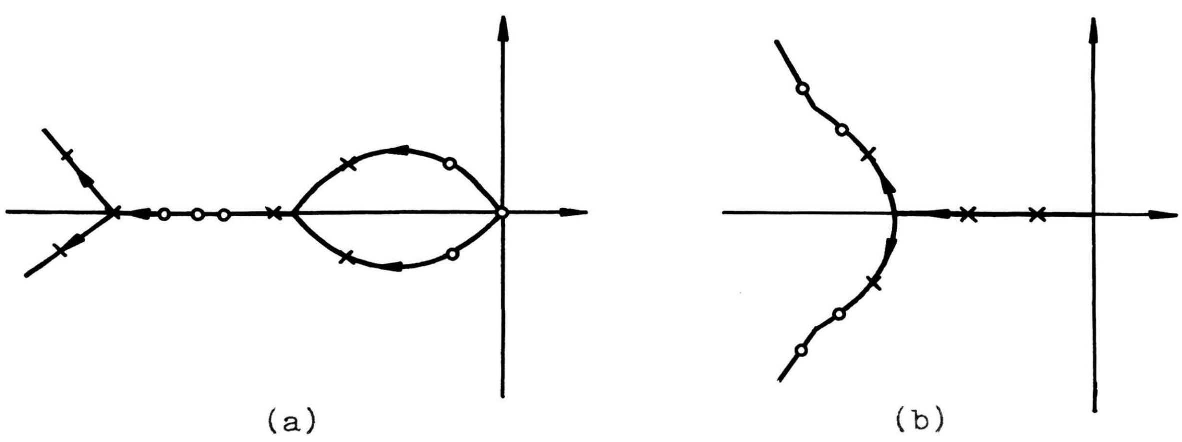


Figure A-2. Examples of
(a) 3-fold alternations and
(b) 4-fold alternations on P-patns

APPENDIX B

CEDERBAUM'S DECOMPOSITION ALGORITHM

Cederbaum (28) has presented a matrix decomposition technique which decomposes a paramount matrix A into BDB', where B is a rectangular unimodular matrix and D is a diagonal matrix. It is noted that the paramount character of the matrix A is a necessary condition for its factorization into the product of BDB'. The decomposition, if possible, is essentially unique. Two definitions are given before the algorithm is presented.

Paramount: A real symmetric matrix is paramount if each principal minor of the matrix is not less than the absolute value of any other minor derived from the same row.

Unimodular: A real rectangular matrix is unimodular if all its subdeterminants (and the elements) equal ± 1 or 0.

The Decomposition Method. Let BDB' be a decomposition of a paramount matrix, $A = (a_{ij})$. If $|a_{pq}|$ is the off-diagonal nonzero element with minimum absolute value, (or one of them or the only one), then in matrix B the row p and q overlap exactly in one column, and $|a_{pq}|$ is equal to one of the elements of the diagonal matrix D.

Let b_1 be the overlap column and $d_1 = |a_{pq}|$ be an element of D which corresponds to the place of b_1 in B. Let

$$A_1 = A - b_1 d_1 b_1'$$

The number of zero, off-diagonal elements of the matrix A_1 is greater than that of the matrix A. Repeat the procedure described above leading to a diagonal matrix A_t,

$$A_2 = A_1 - b_2 d_2 b_2'$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_t = A_{t-1} - b_t d_t b_t'$$

The number of steps t is finite and

$$t \leq n(n-1)/2$$

where n is the order of A.

The procedure is illustrated below with a 3^{rd}-order example. It can be extended to matrices of higher orders. Consider the paramount matrix,

$$A = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 8 & 4 \\ 0 & 4 & 4 \end{bmatrix}$$

Since $a_{12} = 3$ is the smallest nonzero, off-diagonal element, $d_1 = 3$, and

$$b_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Then

$$b_1 d_1 b_1' = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$A_1 = A - b_1 d_1 b_1'$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 4 \end{bmatrix}$$

The smallest nonzero, off-diagonal element of A_1 is $a_{23}=4$. Thus, $d_2=4$ and

$$b_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Then

$$A_2 = A_1 - b_2 d_2 b_2'$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is diagonal with $d_3=2$ and $d_4=1$. Therefore,

$$b_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$b_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The decomposition is complete, and the two matrices B and D are as shown below,

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

APPENDIX C

ROUTH ARRAY AND TRIDIAGONAL MATRIX

Many authors have been interested in the close association between ladder networks and tridiagonal matrices. This appendix is concerned with presenting the Routh array method of obtaining a tridiagonal matrix from a strictly Hurwitz polynomial (22). It is stated in the form of a theorem.

<u>Theorem</u>: Let the n roots of

$$f(s) = s^n + a_1 s^{n-1} + \ldots + a_n$$

have negative real parts. Then there exists a matrix F,

$$F = \begin{bmatrix} -k_1 & -k_2 & & & & \bigcirc \\ k_2 & 0 & -k_3 & & & \\ & k_3 & 0 & -k_4 & & \\ & & \ddots & \ddots & \ddots & \\ & \bigcirc & & \ddots & \ddots & -k_n \\ & & & & k_n & 0 \end{bmatrix}$$

having f(s) as its characteristic polynomial where the entries k_i, $i = 1, 2, \ldots, n$ are found from the first column of the modified Routh array as follows;

$$
\begin{array}{llll}
1 & a_2 & a_4 & a_6 \cdots \\
\boxed{k_1 = a_1 \qquad a_3 \qquad a_5 \qquad a_7 \cdots} & & & \\
1 & a_3/a_1 & a_5/a_1 & a_7/a_1 \cdots \\
\boxed{k_2^2 = a_2 - a_3/a_1 \qquad a_4 - a_5/a_1 \qquad a_6 - a_7/a_1 \cdots} & & & \\
1 & (a_4 - a_5/a_1)/k_2^2 & (a_6 - a_7/a_1)/k_2^2 \cdots & \\
\boxed{k_3^2 = a_3/a_1 - (a_4 - a_5/a_1)/k_2^2 \cdots} & & & \\
1 & \cdots & &
\end{array}
$$

Note that f(s) has roots with negative real parts; therefore, the entries in the first column of the Routh array are greater than zero. By dividing by a positive number, the positiveness of the coefficients in the first column of the array is not altered.

APPENDIX D

THE SIMILARITY TRANSFORMATION T_n

The transformation matrix T_n in equation (4.B.13) is a similarity transformation such that

$$T_n^{-1} A_n T_n = F_n \tag{D.1}$$

or

$$A_n T_n = T_n F_n \tag{D.2}$$

where A_n is a companion matrix as shown in equation (4.B.3) and F_n is a tridiagonal matrix as shown in Appendix C. It is proved by mathematical induction as follows.

First consider when n = 2. From the left-hand side of equation (D.2),

$$A_2T_2 = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} 0 & 1/k_2 \\ 1 & 0 \end{bmatrix}$$

$$A_2T_2 = \begin{bmatrix} 1 & 0 \\ -a_1 & -a_2/k_2 \end{bmatrix} \tag{D.3}$$

From the right-hand side of equation (D.2)

$$T_2F_2 = \begin{bmatrix} 0 & 1/k_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -k_1 & -k_2 \\ k_2 & 0 \end{bmatrix}$$

$$T_2F_2 = \begin{bmatrix} 1 & 0 \\ -k_1 & -k_2 \end{bmatrix} \tag{D.4}$$

From the Routh array we have $k_1 = a_1$ and $k_2^2 = a_2$. Thus, equation (D.3) and equation (D.4) are equal and T_n is valid for $n = 2$.

Next consider when $n = 3$. From the left-hand side of equation (D.2) we have

$$A_3T_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1/k_2k_3 \\ 0 & 1/k_2 & 0 \\ 1 & 0 & -k_3/k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/k_2 & 0 \\ 1 & 0 & -k_3/k_2 \\ -a_1 & -\dfrac{a_2}{k_2} & \left(\dfrac{-a_3}{k_2k_3} + \dfrac{a_1k_3}{k_2}\right) \end{bmatrix} \tag{D.5}$$

From the right-hand side of equation (D.2) we have

$$T_3F_3 = \begin{bmatrix} 0 & 0 & 1/k_2k_3 \\ 0 & 1/k_2 & 0 \\ 1 & 0 & -k_3/k_2 \end{bmatrix} \begin{bmatrix} -k_1 & -k_2 & 0 \\ k_2 & 0 & -k_3 \\ 0 & k_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/k_2 & 0 \\ 1 & 0 & -k_3/k_2 \\ -k_1 & -k_2 - \dfrac{k_3^2}{k_2} & 0 \end{bmatrix} \tag{D.6}$$

From the Routh array we have

$$k_1 = a_1$$

$$k_2^2 = a_2 - a_3/a_1$$

and

$$k_3^2 = a_3/a_1$$

It is verified that equation (D.5) is equal to equation (D.6). Therefore, T_n is true for n = 3.

Assume T_n is true for an arbitrary order n. In the following we prove that it is true for the order (n+1), such that

$$T_{n+1}^{-1} A_{n+1} T_{n+1} = F_{n+1} \tag{D.7}$$

or

$$A_{n+1} T_{n+1} = T_{n+1} F_{n+1} \tag{D.8}$$

The resulting matrices of the left and right sides of equation (D.8) are equations (D.9) and (D.10), respectively.

$$A_{n+1}T_{n+1} = A_{n+1}P_{n+1}K_{n+1}^{-1} =$$

$$\begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & & & & \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ \cdot & & & & & \cdot & \\ 0 & & & & & & 1 \\ -a_{n+1} & -a_n & & \cdot & \cdot & \cdot & -a_1 \end{bmatrix} \begin{bmatrix} & & & & & & 1 \\ & & & & & 1 & 0 \\ & \bigcirc & & & 1 & 0 & -k_{n+1}^2 \\ & & & \cdot & & -(k_n^2+k_{n+1}^2) & 0 \\ & & \cdot & & \cdot & & \cdot \\ & 1 & \cdot & \cdot & & & \cdot \\ 1 & 0 & \cdot & & & & p_{n,n+1} \\ 1 & 0 & -(k_3^2+\cdots+k_{n+1}^2) & \cdots & & p_{n+1,n} & p_{n+1,n+1} \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & \frac{1}{k_2} & & & \bigcirc \\ & & \cdot & & \\ & & & \cdot & \\ & \bigcirc & & & \cdot \\ & & & & \frac{1}{k_2\cdots k_{n+1}} \end{bmatrix}$$

$$
A_{n+1}T_{n+1}=\left[\begin{array}{cccccccc}
 & & & & & & \frac{1}{k_2k_3\cdots k_n} & 0 \\
 & \bigcirc & & & & \frac{1}{k_2k_3\cdots k_{n-1}} & 0 & \frac{-k_{n+1}}{k_2k_3\cdots k_n} \\
 & & & & \cdot & 0 & \frac{-(k_n^2+k_{n+1}^2)}{k_2k_3\cdots k_n} & 0 \\
 & & & \cdot & 0 & \frac{-(k_{n-1}^2+k_n^2+k_{n+1}^2)}{k_2k_3\cdots k_{n-1}} & 0 & \frac{k_{n+1}(k_n^2+k_{n+1}^2)}{k_2k_3\cdots k_n} \\
 & \frac{1}{k_2} & \cdot & \cdot & \cdot & & & \vdots \\
1 & 0 & \frac{-(k_3^2+\cdots+k_{n+1}^2)}{k_2k_3} & \cdots & & & \frac{p_{n+1,n}}{k_2\cdots k_n} & \frac{p_{n+1,n+1}}{k_2\cdots k_{n+1}} \\
-a_1 & \frac{-a_2}{k_2} & \frac{-a_3+a_1(k_3^2+\cdots+k_{n+1}^2)}{k_2k_3} & & \cdots & & & (n+1,n+1)
\end{array}\right] \qquad (D.9)
$$

$$T_{n+1}F_{n+1} = P_{n+1}K_{n+1}^{-1}F_{n+1} = P_{n+1}\,(K_{n+1}^{-1}F_{n+1}) =$$

$$\begin{bmatrix}
 & & & & & 1 \\
 & \bigcirc & & & 1 & 0 \\
 & & & 1 & 0 & -k_{n+1}^2 \\
 & & \cdot & \cdot & \cdot & \vdots \\
 & \cdot & \cdot & \cdot & \cdot & \vdots \\
 & 1 & 0 & -(k_4^2+\ldots+\ldots k_{n+1}^2) & \cdots & p_{n,n+1} \\
1 & 0 & -(k_3^2+\ldots+k_{n+1}^2) & \cdots & & p_{n+1,n+1}
\end{bmatrix}
\begin{bmatrix}
-k_1 & -k_2 & & & & \\
1 & 0 & -\frac{k_3}{k_2} & & \bigcirc & \\
 & \frac{1}{k_2} & 0 & -\frac{k_4}{k_2k_3} & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & 0 & -\frac{k_n}{k_2\ldots k_{n-1}} & \\
 & \bigcirc & & \frac{1}{k_2\ldots k_{n-1}} & 0 & -\frac{k_{n+1}}{k_2\ldots k_n} \\
 & & & & \frac{1}{k_2\ldots k_n} & 0
\end{bmatrix}$$

$$
T_{n+1}F_{n+1}=\begin{bmatrix}
 & & & & \frac{1}{k_2k_3\cdots k_n} & 0 \\
 & \bigcirc & & \frac{1}{k_2k_3\cdots k_{n-1}} & 0 & \frac{-k_{n+1}}{k_2k_3\cdots k_n} \\
 & & & 0 & \frac{-(k_n^2+k_{n+1}^2)}{k_2k_3\cdots k_n} & 0 \\
 & & \cdot & & & \\
 & \cdot & \cdot & \frac{-(k_{n-1}^2+k_n^2+k_{n+1}^2)}{k_2k_3\cdots k_{n-1}} & 0 & \frac{k_{n+1}(k_n^2+k_{n+1}^2)}{k_2k_3\cdots k_n} \\
 & \frac{1}{k_2} & \cdot & & & \vdots \\
1 & 0 & \frac{-(k_3^2+\cdots+k_{n+1}^2)}{k_2k_3} & \cdots & \frac{p_{n,n+1}-k_n^2p_{n,n-1}}{k_2k_3\cdots k_n} & \frac{-k_{n+1}p_{n,n}}{k_2\cdots k_n} \\
-k_1 & -k_2-\left(\frac{k_3^2+\cdots+k_{n+1}^2}{k_2}\right) & 0 & \cdot & \cdot\quad\cdot & \frac{-k_{n+1}p_{n+1,n}}{k_2k_3\cdots k_n}
\end{bmatrix}
\tag{D.10}
$$

The (n+1,n+1) term in equation (D.9) is

$$\frac{-1}{k_2 \cdots k_{n+1}}(a_{n+1} + a_{n-1}p_{3,n+1} + a_{n-3}p_{5,n+1} + \cdots + a_1 p_{n+1,n+1})$$

In order to prove that equations (D.9) and (D.10) are equal, we have to use a set of relations from the n^{th}-order case. From equation (D.2),

$$A_n T_n = T_n F_n \tag{D.2}$$

the resulting matrices are shown in equation (D.11). By equating like entries of equation (D.11) we obtain a set of relations as follows:

$$\begin{aligned}
a_1 &= k_1 \\
a_2 &= k_2^2 + k_3^2 + \cdots + k_n^2 \\
a_3/a_1 &= k_3^2 + k_4^2 + \cdots + k_n^2 \\
&\cdot \quad \cdot \quad \cdot \\
p_{n,n-1} &= p_{n-1,n} - k_{n-1}^2 \, p_{n-1,n-2} \\
p_{n,n} &= -k_n^2 \, p_{n-1,n-1}
\end{aligned}$$

$$a_n + a_{n-2}p_{3,n} + a_{n-4}p_{5,n} + \cdots + a_1 p_{n,n} = k_n^2 \, p_{n,n-1}$$

This set of relations is valid for an arbitrary order n and can be verified by using the Routh array and equation (4.B.16b).

$$A_nT_n = T_nF_n \qquad \text{(D.2)}$$

or

$$\begin{bmatrix} & & & & \frac{1}{k_2 \cdot\cdot k_{n-1}} & 0 \\ & \bigcirc & & \frac{1}{k_2 \cdot\cdot k_{n-2}} & 0 & \frac{-k_n}{k_2 \cdot\cdot k_{n-1}} \\ & & \cdot^{\cdot^{\cdot}} & & & \vdots \\ & \frac{1}{k_2} & & & & \\ 1 & 0 & \cdot\;\cdot\;\cdot & & \frac{p_{n,n-1}}{k_2 \cdot\cdot k_{n-1}} & (n-1,n) \\ -a_1 & -\frac{a_2}{k_2} & \frac{-a_3+a_1(k_3^2+\cdot\cdot+k_n^2)}{k_2k_3} & \cdot\;\cdot\;\cdot & & (n,n) \end{bmatrix} = \begin{bmatrix} & & & & \frac{1}{k_2 \cdot\cdot k_{n-1}} & 0 \\ & \bigcirc & & \frac{1}{k_2 \cdot\cdot k_{n=2}} & 0 & \frac{-k_n}{k_2 \cdot\cdot k_{n-1}} \\ & & \cdot^{\cdot^{\cdot}} & & & \vdots \\ & \frac{1}{k_2} & & & & \\ 1 & 0 & & & & \frac{-k_n p_{n-1,n-1}}{k_2 \cdot\cdot k_{n-1}} \\ -k_1 & -k_2-\left(\frac{k_3^2+\cdot\cdot k_n^2}{k_2}\right) & 0 & \cdot\;\cdot\;\cdot & & \frac{-k_n p_{n,n-1}}{k_2 \cdot\cdot k_{n-1}} \end{bmatrix}$$

(D.11)

where $(n-1,n) = (p_{n,n-1})/k_2 \cdot\cdot k_n$ and

$$(n,n) = -(a_n + a_{n-2}p_{3,n} + \ldots + a_1 p_{n,n})/k_2 \ldots k_n$$

For the case of order (n+1), this set of relations is

$$a_1 = k_1$$

$$a_2 = k_2^2 + k_3^2 + \dots + k_{n+1}^2$$

$$a_3/a_1 = k_3^2 + \dots + k_{n+1}^2$$

$$\cdot \quad \cdot \quad \cdot \tag{D.13}$$

$$p_{n+1,n} = p_{n,n+1} - k_n^2 \, p_{n,n-1}$$

$$p_{n+1,n+1} = -k_{n+1}^2 \, p_{n,n}$$

$$a_{n+1} + a_{n-1}p_{3,n+1} + a_{n-3}p_{5,n+1} + \dots + a_1 p_{n+1,n+1}$$

$$= k_{n+1}^2 \, p_{n+1,n}$$

Substituting the above relations into equation (D.9), we obtain equation (D.10). Therefore, the similarity transformation is valid for the order (n+1) and the proof is complete.

APPENDIX E

GYRATOR AS A NETWORK ELEMENT

An ideal gyrator is a passive lossless element, because the sum of the input powers to its two ports is zero. Practically, its realization requires the use of active devices. Therefore, some authors consider it as an active element. In the following, a brief description of the realization of an ideal gyrator and its application are given for easy reference.

1. Circuit Realization of An Ideal Gyrator.

The admittance matrix of an ideal gyrator can be expressed as the sum of two matrices;

$$Y_G = \begin{bmatrix} 0 & g \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -g & 0 \end{bmatrix} \tag{E.1}$$

By denoting

$$Y_a = \begin{bmatrix} 0 & g \\ 0 & 0 \end{bmatrix} \tag{E.2}$$

and

$$Y_b = \begin{bmatrix} 0 & 0 \\ -g & 0 \end{bmatrix} \tag{E.3}$$

the realization of the ideal gyrator is achieved by realizing Y_a and Y_b respectively, and then connecting them in parallel, as shown in Figure E-1. Both Y_a and Y_b are admittance matrices of voltage-controlled current sources.

A transistor realization of the gyrator can be obtained from the controlled-source representation of Figure E-1. One such transistor circuit is shown in Figure E-2 (6). A detailed analysis can be found in (15). Experimental data of this circuit shows excellent performance over wide ranges of load and frequencies. An ideal gyrator can also be realized with two operational amplifiers (12), where stability of the circuit has been considered. Chua and Newcomb (11) have described an integrated direct-coupled gyrator containing 1 diode, 12 resistors, and 9 transistors.

2. Applications.

One major use of gyrators is the replacement of inductors and transformers by gyrators and capacitors. The advantages of RC gyrator circuits are (5) :

1) Low sensitivity - this results from the low sensitivity of passive RLC networks.
2) Tunability - the resonating frequencies can be adjusted by varying the gyration conductances or by trimming the capacitances.
3) The integrated gyrator has the advantages of low-cost, high performance and small size.

A gyrator circuit using two operational amplifiers (6) is shown in Figure E-3.

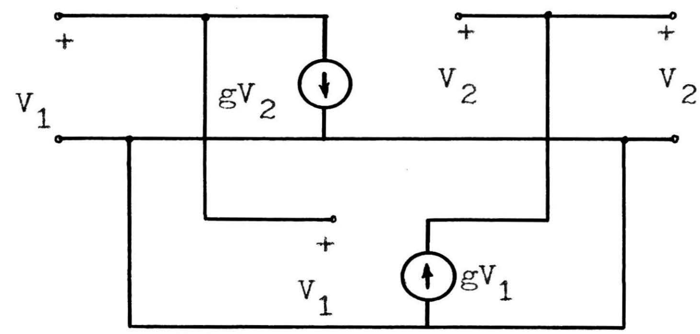


Figure E-1. Controlled-source model of an ideal gyrator

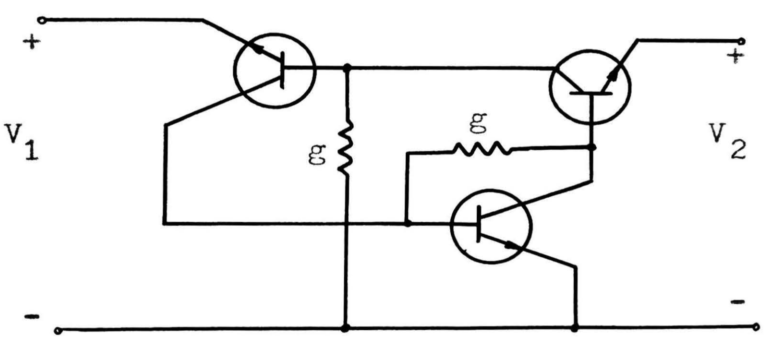
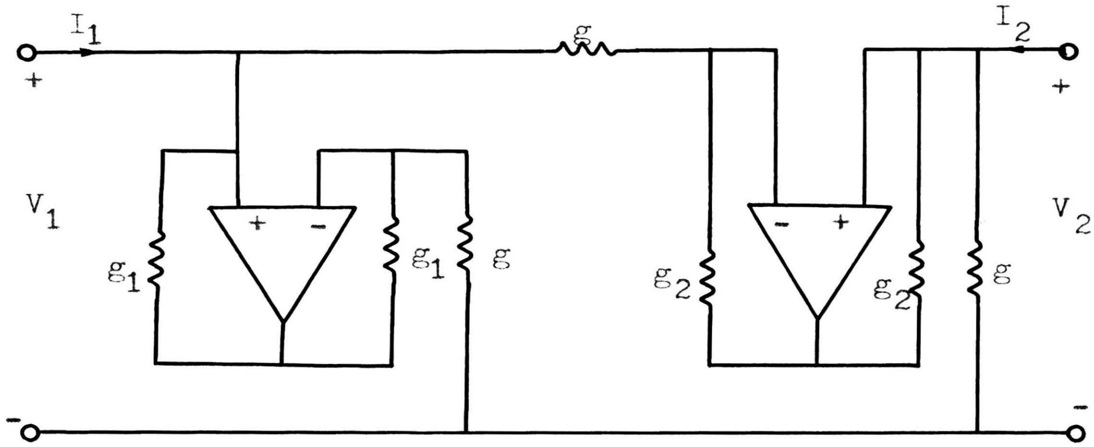


Figure E-2. Three-transistor realization of an ideal gyrator



$$I_1 = -gV_2$$

$$I_2 = gV_1$$

Figure E-3. A gyrator circuit using two operational amplifiers